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Designing weighted and directed networks under complementarities

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ABSTRACT

Strategic complementarities influence various social and economic activities. This study introduces a model to design a weighted and directed complementarity network to achieve a planner's objectives. The network represents the direction and intensity of complementarities between agents, influencing their best-responses to one another and determining equilibrium efforts. The planner's objective function can be convex, as commonly assumed in prior research, or arbitrarily concave to represent scenarios with diminishing marginal returns to each agent's effort. In all scenarios, optimal networks are generalized nested split graphs (GNSGs) which exhibit a link-dominance hierarchy among agents. These optimal networks are often strictly hierarchical, leading to inequality between ex ante identical agents. Additional analysis of a non-cooperative network formation game reveals that all decentralized equilibrium networks are inefficient GNSGs.

1. Introduction

When complementarities are present, an individual's effort or productivity increases with that of others. Such complementarities are prevalent in social and economic organizations and can arise through various channels, such as knowledge spillovers (Arrow, 1962), peer pressure (Kandel and Lazear, 1992), and social preferences (e.g., Sugden, 1984; Rabin, 1993; Fehr and Schmidt, 1999). Considering the prevalence of complementarities and their impact on aggregate outcomes, an organizational planner may want to optimize the structure of complementarities to achieve their objectives.

In some circumstances, the planner can manipulate the *directions* and *intensities* of complementarities. For instance, Mas and Moretti (2009) examine peer effects in traditional workplaces and find that among supermarket workers, "worker effort is positively related to the productivity of workers who see him, but not workers who do not see him." Therefore, the manager (planner) can change the *direction* of complementarities by altering the flow direction of productivity-related information between workers. "[A]dditionally, workers respond more to the presence of coworkers with whom they frequently interact. (Mas and Moretti, 2009)" As a result, the manager can manipulate the *intensity* of complementarities by managing the workers' overlapping working hours.

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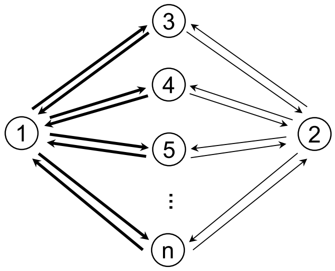

Fig. 1. An illustration of agent 1 link-dominating agent 2. The thickness of links indicates the weights.

This paper examines a planner's problem of designing a weighted and directed network under complementarities. The weight and direction of a link represent the intensity and direction of complementarities, respectively. First, we introduce the new notions of link-dominance and generalized nested split graph (GNSG). Previous literature (König et al., 2014; Belhaj et al., 2016; Hiller, 2017) defines and examines nested split graphs among simple networks, i.e., networks with unweighted and undirected links. GNSGs extend this concept to encompass weighted and directed networks. Second, we demonstrate that all solution networks to the planner's network design problem and all equilibrium networks in a decentralized network formation game are GNSGs (Theorems 1 and 2). Third, while previous studies focus on simple networks and convex objective functions of the planner (e.g., Belhaj et al. (2016) and Hiller (2017)), our analysis allows for general objective functions of the planner and non-linear best-response functions of agents.

Our model involves a planner, a set of homogeneous agents, and two stages.[2] First, the planner chooses a weighted and directed network from a set of feasible networks. The planner's chosen network determines the complementarity technology that influences the agents' interactions in the second stage. Second, each agent chooses an effort level. If a *directed link* exists from agent i to agent j, then j's effort increases with i's effort, and the marginal increment is increasing in the link's weight.[3] Thus, the directions of links represent the directions of complementarities and the weights of links represent their intensities.

Establishing complementarities is resource-intensive; thus, not all networks are feasible. Each link incurs a cost. A network is *feasible* to establish by the planner if and only if the total costs of all links do not surpass the planner's available budget. The planner's problem is to choose a feasible network to maximize an objective function that is increasing in each agent's effort.[4] The solution networks to the planner's problem are referred to as *optimal networks*.

First, we find that all optimal networks are GNSGs, regardless of whether the planner's objective function is convex or strictly concave (Theorem 1). GNSGs are defined using the concept of link-dominance. As illustrated in Fig. 1, agent 1 *link-dominates* agent 2 if, for each of the other agents (let's say i), the weight of the link from 1 to i is greater than that of the link from 2 to i, and the weight of the link from i to 1 is greater than that of the link from i to 2. A GNSG is a weighted and directed network in which, for any two agents, one link-dominates the other. In other words, GNSGs exhibit a total *link-dominance ordering* among agents.[5]

Second, extremely hierarchical GNSGs are optimal in various environments, leading to endogenous inequality. Specifically, we identify a range of environments in which optimal networks are *strict GNSGs* such that all agents are totally and strictly ordered by link-dominance (Proposition 1). These environments include cases where the planner's objective function and the agents' best-response function are arbitrarily convex or modestly concave. In a strict GNSG, no two agents are created equal. Among each pair of agents, one strictly link-dominates the other and exerts strictly greater effort. As long as agents' utilities are monotonic in efforts, inequality arises endogenously between ex-ante identical agents.

Third, all equilibrium networks in a decentralized network formation game are inefficient GNSGs (Theorem 2). In the decentralized network formation game, each agent chooses the weights of the links directly connected to them. The inefficiency of equilibrium networks stems from two channels. The first channel is that with complementarities, each agent's links generate positive externalities on other agents, but these externalities are overlooked by individual agents. As a result, the weight of each link chosen by agents is smaller than the level that maximizes the agents' total utilities. The second channel is the agents' failure to coordinate on a *strictly* hierarchical GNSG: all networks maximizing agents' total utilities with the same costs as an equilibrium network are strictly hierarchical, but some equilibrium networks are not.

[2] We examine the case of heterogeneous agents in our Online Appendix.

[3] For the sake of clarity, we refer to the endogenous outcome variable as each agent's *effort* throughout this paper. However, one can consider the complementarities representing knowledge spillovers and the outcome variable as each agent's productivity.

[4] In our Online Appendix, we examine the situation where the planner's objective function is decreasing in each agent's effort.

[5] In a simple nested split graph, a list of centrality measures, including degree centrality and eigenvector centrality, induce the same ordering of agents (König et al., 2014) as that induced by link-dominance. We discuss in Section 2.3 that this property does not generally hold for GNSGs.

1.1. Literature

Our study contributes to the burgeoning literature on the network approach for examining the formation and optimal design of social and economic organizations. Specifically, we combine i) the network game approach to model the directions and intensities of complementarities (Ballester et al., 2006; Acemoglu et al., 2012; Bramoullé et al., 2014) and ii) the analysis of network design problems from a planner's perspective (Corbo et al., 2006; Belhaj et al., 2016; Hiller, 2017).

Our work is closely related to Corbo et al. (2006); Belhaj et al. (2016); Hiller (2017); König et al. (2014), who explore optimal networks under complementarities from a planner's perspective but limit their analysis to simple networks. From these previous studies, it is known that among simple networks, the networks maximizing a convex objective function of the planner are simple nested split graphs. However, three questions remain: (i) how to generalize simple nested split graphs to weighted and directed networks; (ii) whether optimal weighted and directed networks exhibit analogous nestedness and hierarchy properties; and (iii) what occurs when the planner's objective function is strictly concave, as naturally arises in certain production environments.[6] Our study addresses all three questions. Notably, a strictly concave objective function differ qualitatively from a convex one: while a convex objective function represents increasing marginal returns to an agent's effort, a strictly concave objective function represents diminishing marginal returns. Despite this distinction, we show that optimal networks are GNSGs for both convex and strictly concave objective functions.

Recent studies on endogenous weighted networks include Bloch and Dutta (2009); Olaizola and Valenciano (2020); Salonen (2016); Baumann (2021). These studies differ from ours in a key respect: they do not consider complementarities in individual activities. In the models of Bloch and Dutta (2009), Salonen (2016) and Olaizola and Valenciano (2020), agents only form links and do not engage in other individual activities; thus, no complementarity exists between individual activities. Due to the omission of complementarity, they identify different networks from our GNSGs. Despite the labels "(weighted) nested split graph" and "dominant nested split graph," the concepts proposed by Olaizola and Valenciano (2020) are distinct from our GNSGs. They define an undirected network as a (weighted) nested split graph if it can be considered an unweighted nested split graph by *ignoring* the weights of all links. Moreover, as their example (Olaizola and Valenciano, 2020, p. 83, Fig. 3) shows, what they call a dominant nested split graph is generally *not* a GNSG based on our definition. Thus, their solution networks are generally not optimal in our network design problem under complementarities. Salonen (2016) has not considered optimal or efficient networks and his analysis starts with heterogeneous agents. In our model, inequality and asymmetry arise endogenously. In Baumann (2021), an agent's investment in self-activities and the investment in connections with others are *substitutes*.

Similar to our study, Cabrales et al. (2011) assume that network links and individual activities are complementary. However, in their model, each agent chooses a single socialization level for interactions with all others. Thus, the choice set in their model is much more constrained than ours and it directly implies that all networks under consideration—whether stable, efficient, or not—are networks with *symmetric links*. In contrast, we allow an agent's inward and outward links with different agents to vary, and we uncover that optimal networks feature *asymmetric links* in a range of environments.[7]

Our paper is structured as follows. Section 2 introduces the model. Section 3 characterizes optimal networks. Section 4 considers a non-cooperative network formation game to examine the decentralized formation of weighted and directed networks. Section 5 concludes the paper. Appendix A contains the proofs of all formal results presented in the main text. Our Online Appendix examines four extensions: heterogeneous agents; networks that minimize the sum of agents' effort levels; the effects of imposing a weight cap on each link; and the planner choosing the network and the agents' efforts simultanously to maximize aggregate efficiency.

2. The network design problem under complementarities

2.1. The model

Consider a planner and a set of agents, $N = \{1, 2 \ldots, n\}$, with $n \geq 3$. The planner's network design problem is modeled as a two-stage game between the planner and the agents.

Stage 1. The planner chooses a feasible (weighted and directed) network. A network $G \in \mathbb{R}_+^{n\times n}$ is an n-by-n nonnegative matrix with zeros on the main diagonal. The ijth entry, $g_{ij} \geq 0$, is the weight of the directed link leading from agent j to agent i. We refer to g_{ij} as an *outward link of* j and an *inward link* of i. It represents the intensity of the directed complementarity from j to i. If $g_{ij} = 0$, the directed complementary is absent. The value of g_{ij} may differ from g_{ji}. We maintain $g_{ii} = 0$ for each $i \in N$.

Establishing network links is costly; hence, not all networks are feasible. Let $b > 0$ denote the planner's available budget. Let $c(g_{ij})$ denote the costs of establishing link g_{ij}, with $c(0) = 0$, $c' > 0$, and $c'' \geq 0$. A network G is *feasible* if the sum of all links' costs does not exceed the planner's available budget, i.e.,

[6] For instance, Akcigit et al. (2018) discover that knowledge spillovers among patent inventors exhibit diminishing marginal effects.

[7] There is a branch of literature on input-output production networks (see Carvalho and Tahbaz-Salehi (2019) for a review) that also examine weighted and directed networks. However, the players in a production network face very different incentives and constraints from ours – for example, a firm's production choices are constrained by production technologies and the prices and availabilities of products in a market equilibrium.

$$\sum_{i\in N}\sum_{j\in N} c(g_{ij}) \le b. \tag{1}$$

This budget constraint is analogous to the organizational resource constraints in the problems examined by Dessein et al. (2016) and Galeotti et al. (2020). The underlying assumption is that complementarities are not free; they require resources such as organizational attention (Simon, 1971) and time and space for organizational members to interact and communicate. These resources are limited at the organizational level.

Stage 2. Given the complementary technology described by G, each agent $i \in N$ simultaneously chooses an effort level $x_i \ge 0$. Let $\mathbf{x} = (x_1, x_2, \dots, x_n)$ denote the effort profile and x_{-i} denote the effort profile of agents other than i. Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be strictly increasing and twice-differentiable. Given $\mathbf{x}$ and G, the payoffs for each agent $i \in N$ are represented by the utility function

$$u_i(\mathbf{x}, G) = \phi\left(\sum_{j\in N} g_{ij}x_j\right) x_i - \frac{1}{2}x_i^2. \tag{2}$$

The first term $\phi(\sum_j g_{ij}x_j)x_i$ represents the private return to i's effort, with $\phi(\sum_j g_{ij}x_j)$ being the marginal benefit for exerting x_i. The second term $\frac{1}{2}x_i^2$ denotes the costs. If $g_{ij} = 0$ for each $j \in N$, then the marginal benefit for exerting efforts would be $\phi(0)$, which we normalize to $\phi(0) = 1$.[8] If $g_{ij} > 0$, then the marginal benefit for exerting efforts $\phi(\sum_j g_{ij}x_j)$ increases with j's effort x_j. This positive effect reflects the directed complementarities from j to i. The function ϕ can be concave or convex, depending on the application at hand. When ϕ is strictly concave (convex), complementarities exhibit diminishing (increasing) marginal effects.

Taking the derivative of $u_i(\mathbf{x}, G)$ with respect to x_i, we obtain the agents' *best-response function* in the second stage:

$$x_i = \phi\left(\sum_{j\in N} g_{ij}x_j\right). \tag{3}$$

We assume the following so that a unique equilibrium effort profile $\mathbf{x}$ exists for each feasible network. Let $\|G\| = (\sum_i\sum_j g_{ij}^2)^{1/2}$.

Assumption 1. There is a bound $\lambda > 0$ on ϕ' such that $\phi'(z) \le \lambda$ for each $z \ge 0$, and $\lambda\|G\| \le 1$ for each feasible G.[9]

Assumption 1 limits the agents' sensitivity to each other by requiring that $\frac{\partial x_i}{\partial x_j} = \phi' g_{ij}$ not be too large for each i, j pair. If this were not the case, agents' efforts could amplify each other without bound, making an equilibrium effort profile non-existent. Similar assumptions are commonly found in the network game literature, such as in Ballester et al. (2006); Belhaj et al. (2016); Hiller (2017); Baetz (2015) and Galeotti et al. (2020).[10] When the constraint (1) is linear ($\sum_i\sum_j g_{ij} \le b$) or quadratic ($\sum_i\sum_j g_{ij}^2 \le b$), Assumption 1 reduces to the simple condition of $\lambda b < 1$.[11]

Assumption 1 has not imposed any restrictions on the distribution of link weights across agents. Symmetric networks, asymmetric networks, hierarchical networks, and non-hierarchical networks are all feasible. For instance, a non-hierarchical and symmetric complete network with $g_{ij} = c^{-1}(\frac{b}{n(n-1)})$ for each link is feasible, as is a hierarchical and asymmetric outward star network with $g_{i1} = c^{-1}(\frac{b}{n-1})$ for each $i \ne 1$ and zero weights for all other links.

Lemma 1. *If Assumption 1 holds, then for each feasible network G, a unique $\mathbf{x}(G) \in \mathbb{R}^n_+$ exists such that it constitutes an equilibrium effort profile in the second stage and $\mathbf{x}(G)$ is continuously differentiable.*

Let $f(x_i)$, with $f' > 0$, denote agent i's contribution to the planner's objective. For instance, f is a production function that converts the agent's effort into outputs that directly concern the planner. Then, given the agents' equilibrium effort profile $\mathbf{x}(G)$, the planner's payoffs from network G are given by the *objective function*

$$\pi(G) = \sum_{i\in N} f(x_i(G)). \tag{4}$$

8 Our Online Appendix considers heterogeneous agents.

9 Assumption 1 is a sufficient condition. We can instead impose the following weaker, but more difficult to verify, assumption: for each feasible network G the best-response mapping $\Phi(\cdot, G)$ defined by $\Phi(\mathbf{x}, G) = (\phi(\sum_i g_{1i}x_i), \phi(\sum_i g_{2i}x_i), \dots, \phi(\sum_i g_{ni}x_i))$ is a contraction mapping. If ϕ is linear, then this contraction condition holds if and only if λ times the largest eigenvalue of G is less than 1 for each feasible G, which is the usual assumption for network games, e.g., in Ballester et al. (2006).

10 An alternative approach involves imposing a bound on each agent's effort. This approach can lead to multiple equilibria, as discussed by Bramoullé et al. (2014) and Belhaj et al. (2014). This possibility is further discussed in the concluding section.

11 This observation follows from $\sum_i\sum_j g_{ij}^2 \le (\sum_i\sum_j g_{ij})^2 \le b^2$, with the two inequalities binding when $g_{ij} = b$ for some i and j.

The planner's network design problem is the following.

The planner's problem. Given Assumption 1, the planner aims to choose a network G to maximize $\pi(G) = \sum_{i\in N} f(x_i(G))$ subject to $\sum_{i\in N}\sum_{j\in N} c(g_{ij}) \le b$.[12,13]

We call the solution networks to the planner's problem *optimal networks*.

Lemma 2. *If Assumption 1 holds, then an optimal network exists.*

Given Assumption 1, the planner's problem is well-defined. In subsequent analyses, we maintain Assumption 1 without referring to it repetitively.

2.2. Remarks

We offer three remarks. First, the best-response function (3) is the starting point of our formal analysis. Our results are applicable to all simultaneous-move games in the second stage with a best-response function representable by (3). For instance, we could assume that agents aim to maximize $v_i(\mathbf{x}, G) = 2\sqrt{\phi(\sum_j g_{ij}x_j)x_i} - x_i$, where the first benefit term is strictly concave in x_i and the cost term is linear. This utility function results in the same best-response function $x_i = \phi(\sum_j g_{ij}x_j)$, and all our subsequent results are applicable.

Second, prior research (Corbo et al., 2006; Belhaj et al., 2016; Hiller, 2017; König et al., 2014) assumes that the planner's objective function is convex in each agent's effort. We permit the contribution function f to be convex or strictly concave, thereby covering a broader range of environments. For instance, consider a production environment where the planner's objective function represents total production and $f(x_i)$ denotes agent i's output. As is commonly assumed, production may exhibit diminishing marginal returns, making $f(x_i)$ strictly concave. Moreover, a strictly concave f implies that the planner assigns more decision weight to agents exerting less effort. This characteristic can model the planner's aversion to effort differences among agents – a preference for equality.

Third, in Belhaj et al. (2016), the planner's problem is formulated (using our notations) as choosing G to maximize the unconstrained objective function $\sum_i f(x_i(G)) - \sum_i\sum_j c(g_{ij})$. In this case, link cost is directly incorporated into the planner's objective function without any additional constraints. Our characterization of optimal networks for the constrained optimization problem (i.e., maximizing $\sum_i f(x_i(G))$ subject to $\sum_i\sum_j c(g_{ij}) \le b$) is applicable to this unconstrained optimization problem. To demonstrate this, consider the auxiliary problem of choosing G *and* a number $b \ge 0$ to maximize $\sum_i f(x_i(G)) - b$ subject to $\sum_i\sum_j c(g_{ij}) = b$. Note that G^* maximizes $\sum_i f(x_i(G)) - \sum_i\sum_j c(g_{ij})$ if and only if (G^*, b^*), with $b^* = \sum_i\sum_j c(g^*_{ij})$, solves the auxiliary problem. The latter holds if and only if G^* maximizes $\sum_i f(x_i(G))$ subject to $\sum_i\sum_j c(g_{ij}) \le b^*$ – the problem we consider.[14]

2.3. Generalized nested split graphs

We introduce the notion of link-dominance to define a GNSG.

Link-dominance. Agent i *link-dominates* agent j in a network G, denoted by $i \succeq_G j$, if $g_{ki} \ge g_{kj}$ and $g_{ik} \ge g_{jk}$ for each $k \in N, k \ne i, j$. If the inequalities are all strict, then we say that i *strictly link-dominates* j and it is denoted by $i \succ_G j$.

Note the qualifier $k \ne i, j$ in the definition of link-dominance. When we determine whether an agent link-dominates another, we do not compare the links directly connecting them.[15] Link-dominance generalizes nestedness of neighborhoods in simple networks to general weighted and directed networks: if agent i link-dominates agent j and j has an inward (outward) link with agent k, then i must also have an inward (outward) link with k, and $g_{ik} \ge g_{jk}$ ($g_{ki} \ge g_{kj}$). Based on the concept of link-dominance, we now define GNSGs. When restricted to simple (i.e., unweighted and undirected) networks, the concept of GNSG reduces to the concept of nested split graphs (König et al., 2014).

Generalized nested split graph. A (weighted and directed) network G is a *generalized nested split graph (GNSG)* if, for each $i, j \in N, i \ne j$, we have either $i \succeq_G j$ or $j \succeq_G i$. The network is a *strict GNSG* if for each $i, j \in N, i \ne j$, either $i \succ_G j$ or $j \succ_G i$.

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[12] The planner's objective function being additively separable is not necessary. If $\pi(G)$ maps one-to-one to $\hat{\pi}(G)$ via a strictly increasing transformation, then maximizing $\pi(G)$ is equivalent to maximizing $\hat{\pi}(G)$. Hence, the analysis applies to, for example, maximizing $\hat{\pi}(G) = \Pi_i x_i(G)$, as it is equivalent to maximizing $\pi(G) = \sum_i \ln x_i(G)$.

[13] Our Online Appendix, Section C, examines optimal networks under linear link cost with a link cap $\bar{g}$ such that $g_{ij} \le \bar{g}$ for each i, j. For each optimal network in that case, there is a GNSG that achieves the same optimal outcome.

[14] Since f is increasing, the inequality $\sum_i\sum_j c(g_{ij}) \le b^*$ is binding at the optimum.

[15] Conceptually, when we think of i link-dominating j, we consider that all *inward* links of i are stronger than those of j (in each comparison of $g_{ik} \ge g_{jk}$) and all *outward* links of i stronger than those of j (in each comparison of $g_{ki} \ge g_{kj}$). However, g_{ij} is both an inward link of i and an outward link of j, and g_{ji} is both an outward link of i and an inward link of j. Therefore, to have link-dominance well-defined, we exclude the links between the two agents in focus in the link comparisons.

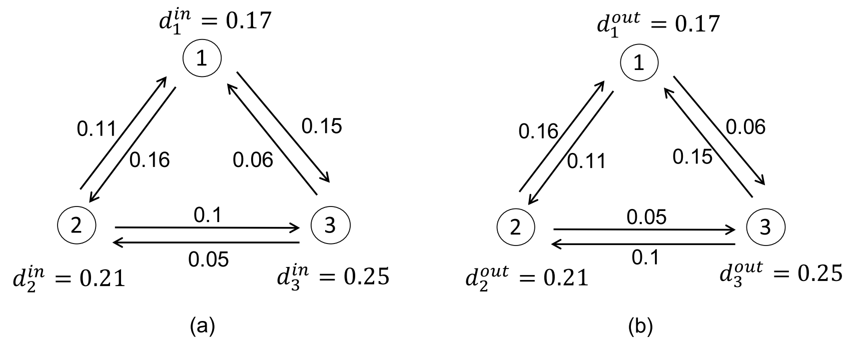

Fig. 2. The number on each link indicates its weight. In-degree is defined by $d_i^{in} = \sum_j g_{ij}$, and out-degree is defined by $d_i^{out} = \sum_j g_{ji}$. In both networks, agent 1 strictly link-dominates agent 2 who strictly link-dominates agent 3. However, we have $d_1^{in} < d_2^{in} < d_3^{in}$ in (a) and $d_1^{out} < d_2^{out} < d_3^{out}$ in (b).

We observe two general properties of GNSGs. First, agents can be totally ordered by link-dominance, and a *nested sequence of cliques* can be constructed. To demonstrate this, consider a GNSG with $1 \succeq_G 2 \succeq_G \cdots \succeq_G n$. Then, we have $g_{12} \geq g_{1i} \geq g_{2i} \geq g_{ji}$ and $g_{21} \geq g_{i1} \geq g_{i2} \geq g_{ij}$ for each $i, j \geq 3$. In other words, the links between the top two agents, 1 and 2, are stronger than those between any other pair of agents. As a result, agents 1 and 2 form a clique. This logic extends to larger groups of agents. For instance, we can think of agents 1, 2 and 3 together forming a larger clique, because any link among them is stronger than a link between any other pair of agents. Consequently, we obtain a nested sequence of cliques: $\{1, 2\} \subseteq \{1, 2, 3\} \subseteq \cdots \subseteq N$.

Second, link-dominance ordering does not necessarily coincide with the orderings induced by conventional centrality measures in a GNSG with weighted and directed links. In undirected networks, centrality measures often induce the same ordering in a simple nested split graph. As demonstrated by König et al. (2014), degree, Bonacich, closeness, betweenness, and eigenvector centralities induce the same ordering in a simple nested split graph. However, directed networks introduce a distinction between inward and outward measures for degree, closeness, and eigenvector centralities. This distinction can result in different orderings that do not align with link-dominance. Fig. 2 presents an example that contrasts link-dominance with in-degree and out-degree centralities. In both networks presented in the figure agent 1 link-dominates agent 2 who link-dominates agent 3. However, in network (a) agent 3 has a greater in-degree than agent 2 who has a greater in-degree than agent 1, whereas in network (b) agent 3 has a greater out-degree than agent 2 who has a greater out-degree than agent 1. This observation highlights the complexity of weighted and directed networks and shows that link-dominance ordering could provide additional insights not fully captured by traditional centrality measures.[16]

3. Optimal networks

3.1. Marginal value of a link

To establish that all optimal networks are GNSGs, we first need to characterize the marginal value of a link. Consider a fixed feasible network G. Let $x_i = x_i(G)$. The *marginal value of link g_{ij} in G* is defined by

$$MV_{ij}(G) \equiv \frac{\partial \pi(G)}{\partial g_{ij}} = \sum_{k \in N} f'(x_k) \frac{\partial x_k}{\partial g_{ij}}. \tag{5}$$

This marginal value measures the improvement in the planner's objective function in response to a marginal increase in the weight of the link, g_{ij}. It is viewed from the planner's perspective: it synthesizes the complementarity flows in the network and the ultimate contributions of the agents' effort to the planner's objective. The marginal value of a link is important because given (weakly) convex link costs, $MV_{ij}(G) > MV_{k\ell}(G)$ implies $g_{ij} \geq g_{k\ell}$ in an optimal network. We derive a simple formula for MV_{ij} through several steps.

First, let

$$\Phi(\mathbf{x}, G) = (\phi(\sum_{i \in N} g_{1i} x_i), \phi(\sum_{i \in N} g_{2i} x_i), \ldots, \phi(\sum_{i \in N} g_{ni} x_i)).$$

The function $\Phi(\cdot, G)$ is the agents' best-response mapping given G. Let Φ' be the n-by-n derivative matrix of Φ with respect to $\mathbf{x}$. To simplify notation, let $z_i \equiv \sum_j g_{ij} x_j$ denote i's accumulation of spillovers from others, which is referred to as i's *effort*

[16] In Appendix B, we define closeness, eigenvector, Bonacich, and betweenness centralities for weighted and directed networks based on Jackson (2008) and Opsahl et al. (2010). By calculating these measures for the networks shown in Fig. 2, we show that even with the same link-dominance ordering, the inward measures and outward measures of degree, closeness, and eigenvector centralities can induce *opposite* orderings. Thus, for each of these centralities, either the inward or the outward measure differs from the link-dominance ordering.

potential, as it determines i's effort in the network: $x_i = \phi(z_i)$. The ijth entry of Φ' is $\Phi'_{ij} = \phi'(z_i)g_{ij}$. Let I be the n-by-n identity matrix, and write $M \equiv [I - \Phi']^{-1}$.[17] The ijth entry of M is denoted by m_{ij}.

The elements of M have an intuitive interpretation. To illustrate, consider the case where the agents' best-responses are linear, such that $\phi(z_i) = 1 + \lambda z_i$ where $\phi'(z_i) = \lambda > 0$ is constant. In this case, $\Phi' = \lambda G$, and we have

$$M = [I - \Phi']^{-1} = \sum_{k=0}^{\infty} \left(\Phi'\right)^k = \sum_{l=0}^{\infty} \lambda^k G^k. \tag{6}$$

The matrix G keeps track of direct spillovers through paths of length 1, while the kth power of G, for $k \geq 2$, keeps track of indirect spillovers through all paths of length k. Consequently, M sums *all direct and indirect* spillovers from one agent to another, with spillovers of degree k multiplied by the discount factor λ^k. In particular, m_{ij} sums all direct and indirect spillovers from j to i.[18]

Next, define agent i's *aggregate influence* in G by $\alpha_i(G) \equiv \frac{\partial \pi(G)}{\partial z_i}$, which equals

$$\alpha_i(G) = f'(x_i)m_{ii}\phi'(z_i) + \sum_{j \in N \setminus \{i\}} f'(x_j)m_{ji}\phi'(z_i). \tag{7}$$

Agent i's aggregate influence $\alpha_i(G)$ measures agent i's aggregate contributions to the planner's objective through all of i's direct and indirect influences on all agents and on i herself at the margin.

Using the fact that $M\Phi' = M - I$, we can obtain a recursive equation for $\alpha_i(G)$. Let $\alpha_i = \alpha_i(G)$ for each agent in G. Then,

$$\alpha_i = f'(x_i)\phi'(z_i) + \sum_{j \in N \setminus \{i\}} \alpha_j g_{ji} \phi'(z_i). \tag{8}$$

Based on this equation, agent i's aggregate influence is equal to i's direct marginal contributions to the planner's objective plus a weighted sum of other agents' aggregate influences. Hence, aggregate influence can be defined recursively by a weighted sum of neighbors' aggregate influences and determined as a fixed point of the network system in the same way as we determine the equilibrium effort of each agent.

Remark 1. Given Assumption 1, in a feasible network G, the aggregate influence index $\alpha_i(G)$ exists and is unique for each $i \in N$.[19]

Now, we state the formula to compute the marginal value of a link:

Lemma 3. *Let G be a feasible network. Then*

1. $\frac{\partial x_k(G)}{\partial g_{ij}} = m_{ki}\phi'(z_i)x_j(G)$, *and*
2. $MV_{ij}(G) = \alpha_i(G)x_j(G)$.

That is, the marginal value of the link from j to i simply equals j's effort x_j times i's aggregate influence α_i in the network. When the network referred to is clear, we drop the argument and write $MV_{ij} = \alpha_i x_j$.

The formula $MV_{ij} = \alpha_i x_j$ provides practical guidelines for real-world intervention problems beyond the formal problem examined in this study. Assuming constant marginal link cost, the formula suggests two network intervention policies for a planner with limited control over the network or when there is a pre-existing network in place: i) *if the planner can enhance one agent's outward links, she should prioritize enhancing the links of the agent exerting the greatest effort*; ii) *if the planner can enhance one agent's inward links, she should prioritize enhancing the links of the agent with the greatest aggregate influence*. These intervention policies are concerned with local interventions applicable to any pre-existing network. They provide valuable insights for planners aiming to optimize network structures or improve network performance in various contexts, such as facilitating the diffusion of innovations. By following these policies, planners can make targeted network interventions that maximize the benefits derived from limited resources or existing network structures.

[17] Assumption 1 implies the invertibility of $[\Phi' - I]$; see the proof of Lemma 2.

[18] The formula $M = \sum_{k=0}^{\infty} \left(\Phi'\right)^k$ holds for any general best-response function ϕ and feasible network G. Regardless of whether best-responses are linear, the ijth element of M measures the total marginal increment effects of z_j on x_i through all paths by which j can influence i–paths of length 1, length 2, and so on, up to infinity–in the network.

[19] This result follows from the differentiability of $\mathbf{x}(G)$ and the invertibility of $[I - \Phi']$. The former is given by Lemma 1. The latter is shown in the proof of Lemma 1.

3.2. Optimal networks are generalized nested split graphs

With the observation $MV_{ij} = \alpha_i x_j$, we can state our main result:

Theorem 1. *Regardless of whether link costs are linear ($c'' = 0$) or strictly convex ($c'' > 0$) and whether f and ϕ are concave or convex, every optimal network is a GNSG such that $\alpha_i(G) \geq \alpha_j(G)$ if and only if $x_i(G) \geq x_j(G)$, and either condition implies $i \succeq_G j$.*[20]

The theorem states that every optimal network is a GNSG in which agents are totally ordered by link-dominance. Moreover, the orderings of link-dominance, aggregate influence, and equilibrium effort levels coincide in an optimal network. Therefore, the aggregate influence α_i in an optimal network indicates both the ordering of link-dominance and the ordering of the agents' contributions, $f(x_i)$, to the planner's objective.

Under strictly convex link cost, Theorem 1 is a direct consequence of Lemma 4 below. The case of linear link cost is established separately in Proposition 2. The separate treatment is necessary because, with strictly convex link cost it is sufficient to examine the Kuhn-Tucker first-order conditions of optimization and using the fact $MV_{ij} = \alpha_i x_j$. With linear link cost, however, it is necessary to examine the second-order conditions to determine optimal networks.

Lemma 4. *Suppose $c'' > 0$. Let G be an optimal network, $x_i = x_i(G)$, and $\alpha_i = \alpha_i(G)$.*

1. *If $\alpha_i \geq \alpha_j$, then $g_{ik} \geq g_{jk}$ for each $k \neq i, j$, and $x_i \geq x_j$.*
2. *If $x_i \geq x_j$, then $g_{ki} \geq g_{kj}$ for each $k \neq i, j$, and $\alpha_i \geq \alpha_j$.*
3. *If $\alpha_i \geq \alpha_j$, then $i \succeq_G j$.*

Lemma 4 draws two observations. If an agent i is more influential (has a greater α_i) than agent j, then the planner enhances i's inward links to leverage i's influence, thereby making i exerts more effort (having a greater x_i) than j. Conversely, if agent i exerts more effort than j, then the planner enhances i's outward links to improve i's influence on others, which results in i being more influential (having a greater α_i) than j. The two observations, together, imply that agent i exerts more effort than j if and only if i has greater aggregate influence than j, and either condition implies that agent i link-dominates agent j.

3.3. Optimal networks are strictly hierarchical

This subsection shows that optimal networks are often strictly hierarchical, regardless of the planner's objective function being concave or convex.

Proposition 1 below shows that under certain natural conditions, each optimal network is a *strict* GNSG. In a strict GNSG, no agents are created equal: for any two agents, one must strictly link-dominate the other. This means that all agents are totally and strictly ordered by link-dominance, and as a result, all agents differ in aggregate influence and effort levels. Consequently, whenever agents' payoffs are monotone in equilibrium efforts, inequality arises endogenously among the ex-ante identical agents.[21] Let $h(z) \equiv f(\phi(z))$.

Proposition 1. *Suppose $c'' > 0$ and $c'(0) = 0$. Then, there are thresholds $r > 0, s > 0$ and $t > 0$ such that if $h'' \geq -r$, $\phi'' \geq -s$ and $c'' \leq t$, then every optimal network G is a strict GNSG such that $x_i(G) \neq x_j(G)$ for each $i \neq j$.*

The proposition identifies the following conditions for strict GNSGs being optimal: i) a link's marginal cost $c'(g_{ij})$ is zero at $g_{ij} = 0$; ii) the functions f and ϕ are convex or modestly concave; and iii) the convexity of the link cost function $c(g_{ij})$ is bounded so that a link's marginal cost does not increase too rapidly. The condition $c'(0) = 0$ is a typical Inada condition to ensure interior solutions. It implies $g_{ij} > 0$ for each i, j pair in an optimal network. Since it requires that all links are positively weighted, it works in the direction of equalizing agents' effort and aggregate influence. Despite such equalizing force, optimal networks are strict GNSGs.

What drives strict GNSGs being optimal is a *snowball effect*, which results from the convexity in strengthening complementarities between a fixed pair of agents. To illustrate this effect, consider a network with only two agents and suppose that $x_1 = 1 + gx_2$ and $x_2 = 1 + gx_1$ for a parameter $g \in (0, 1)$. In equilibrium, we have $x_1(g) = x_2(g) = \frac{2}{1-g}$. Fig. 3a plots this function. The agents' efforts increase increasingly in g, which demonstrates the convexity. This convexity implies a snowball effect: if two agents are already (weakly) more connected with each other, then the planner finds it optimal to strengthen the links between them further, at the expense of the links among those who are (weakly) less connected. As a result, agents who are (weakly) more influential become *strictly* more influential because their outward links are further enhanced.

[20] If there is some $k \neq i, j$ with $g_{ik} > 0$ or $g_{ki} > 0$, then $i \succeq_G j$ also implies $\alpha_i(G) \geq \alpha_j(G)$ and $x_i(G) \geq x_j(G)$. The case of $i \succeq_G j$ but $\alpha_i(G) < \alpha_j(G)$ will occur only if i and j are isolated from all other agents: $g_{ki} = g_{ik} = g_{kj} = g_{jk} = 0$ for each $k \neq i, j$.

[21] For our utility function (2), the agents' utilities in equilibrium are $u_i(\mathbf{x}(G), G) = \frac{1}{2}x_i^2(G)$.

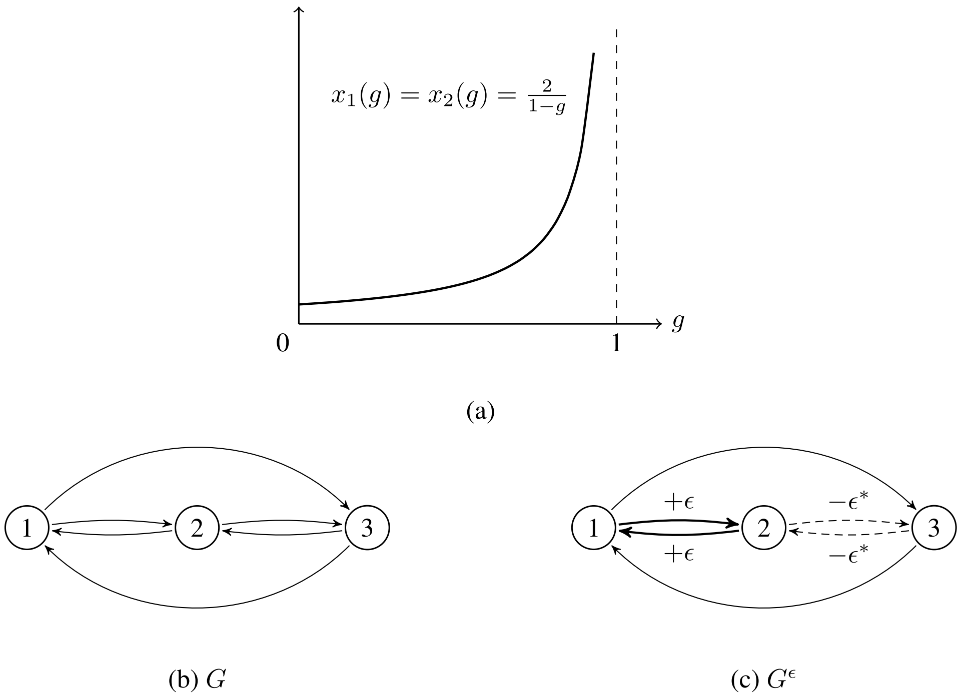

Fig. 3. (a) Convexity in strengthening the complementarities between agents. (b) G is a completely connected network with equally weighted links. It is symmetric everywhere and satisfies the first-order Kuhn-Tucker conditions. However, it is not optimal. (c) G^ϵ is obtained from G by strengthening the links between 1 and 2 by an ϵ amount and weakening the links between 2 and 3 by an ϵ^* amount. The costs of G and G^ϵ are the same but there are positive ϵ and ϵ^* such that $\pi(G^\epsilon) > \pi(G)$.

Agents who exert (weakly) more effort now exert *strictly* more effort because their inward links are further enhanced. Consequently, a strict hierarchy arises.

The following example provides further insight. Consider networks G and G^ϵ shown in Fig. 3. Network G is a complete network with equally weighted links. This network is symmetric in every aspect. Thus, each agent exerts the same effort $x_i(G)$ and has the same aggregate influence $\alpha_i(G)$. However, G is not optimal. From G to G^ϵ, we reallocate resources to strengthen to the links between agent 1 and agent 2 by an $\epsilon > 0$ amount, while simultaneously weakening the links between agents 2 and 3 by an $\epsilon^* > 0$ amount. The values of ϵ and ϵ^* are chosen such that G^ϵ and G have the same costs. If the marginal cost of a link does not increase too rapidly (i.e., $c(g_{ij})$ is not too convex), then $\epsilon^* = \epsilon + o(\epsilon)$ is close to ϵ for small values of ϵ. Consequently, agent 1's effort will increase, and due to the snowball effect, the increase in agent 1's effort will be strictly greater than the reduction in agent 3's effort. Since agent 1's improvement spills over to agent 2, agent 2's effort also increases. Thus, $x_1(G^\epsilon) + x_3(G^\epsilon) \geq x_1(G) + x_3(G)$ and $x_2(G^\epsilon) > x_2(G)$. It follows that if f is not too concave, then we will have $f(x_1(G^\epsilon)) + f(x_2(G^\epsilon)) + f(x_3(G^\epsilon)) > f(x_1(G)) + f(x_2(G)) + f(x_3(G))$. Note that in G^ϵ, agent 1 strictly link-dominates agent 2, who in turn strictly link-dominates agent 3.[22]

However, there are tradeoffs against leveraging the snowball effect. First, convex link cost ($c'' > 0$) implies that $\epsilon^* > \epsilon$. The more convex c is, the more ϵ^* is greater than ϵ, leading to more reduction in the weaker agents' effort. Second, the more concave $h(z) = f(\phi(z))$ is in the planner's objective function, the greater decision weights the planner assigns to weaker agents. Consequently, if c is sufficiently convex and h is sufficiently concave, then the reallocation of weights cannot improve performance. In such scenarios, the non-hierarchical network G can be optimal.

Next, we show that all optimal networks under linear link cost are strictly hierarchical. Without further loss of generality, we assume $c(g_{ij}) = g_{ij}$. Optimal networks under linear link cost take the form of either inward stars or outward quasi-stars, both of which are special cases of GNSGs.

Inward star. An *inward star* is a network G such that after appropriate reindexing, $\sum_{j\in N} g_{1j} > 0$ and $\sum_{j\in N} g_{ij} = 0$ for each $i \neq 1$, i.e., all links are built toward agent 1.

Outward quasi-star. An *outward quasi-star* is a network G such that after appropriate reindexing, i) $g_{12} \geq g_{21} > 0$, ii) $g_{j1} \geq 0$ for each $j \geq 3$, and iii) zero weights for all other links.

[22] This example also demonstrates that examining only the first-order conditions of equalizing $MV_{ij}/c'(g_{ij})$ for each link is insufficient. Given that $x_i(G)$ and $\alpha_i(G)$ are the same across agents, $MV_{ij} = \alpha_i x_j$ is identical for all links in G. Therefore, the condition of equalizing $MV_{ij}/c'(g_{ij})$ holds for such network.

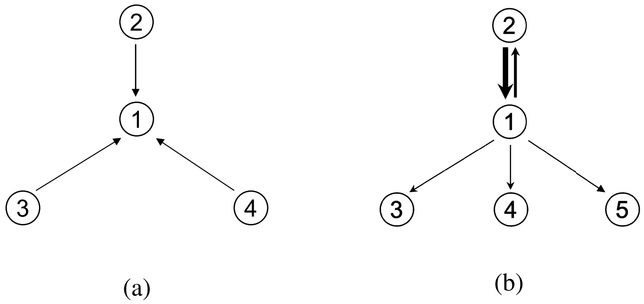

Fig. 4. (a) An inward star. (b) A connected outward quasi-star: $g_{12} > g_{21} > g_{j1}$ for each $j \geq 3$.

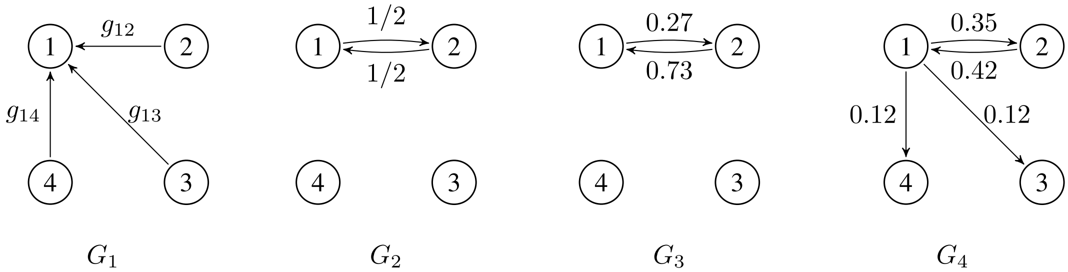

Fig. 5. Examples of optimal networks under linear link cost.

In a feasible network G, let $\bar{x}(G) = \max_{i \in N} x_i(G)$ denote the maximum effort and $\underline{x}(G) = \min_{i \in N} x_i(G)$ denote the minimum effort. Observe that $\bar{x}(G) = x_1(G) > \underline{x}(G)$ in both an inward star and an outward quasi-star.

Proposition 2. *Suppose $c(g_{ij}) = g_{ij}$. Then optimal networks are either all inward stars or all outward quasi-stars. In either case, $\bar{x}(G) > \underline{x}(G)$ for every optimal G.*

The optimality of inward stars versus outward quasi-stars depends on the curvature of the contribution function f and the overall strength of complementarities. If the overall strength of complementarities is weak (i.e., ϕ' is small) and f is highly convex, causing the planner to focus on enhancing the maximum effort across all agents, then optimal networks take the form of inward stars. In all other cases, optimal networks are outward quasi-stars. If f is sufficiently concave, then each optimal network is a connected outward quasi-star, as illustrated in Fig. 4b: two agents form a clique at the center, exerting a strong influence on each other, while one of them imposes a one-way influence on all remaining agents. This design leads to an extreme concentration of influence regardless of the concavity of f. The subsequent remark provides a sufficient condition for the optimality of inward stars versus outward quasi-stars, accompanied by a numerical example. Its proof can be found in Appendix A.

Remark 2. Suppose $c(g_{ij}) = g_{ij}$ and $b = 1$. Consider linear best-responses such that $\phi(z) = 1 + \lambda z$ with $0 < \phi'(z) = \lambda < 1$, and suppose $f'' = \eta$ is a constant. Then there are a threshold $r > 0$ and a strictly increasing and positive function of λ, $D(\lambda) > 0$, such that

1. if $\phi'(z) = \lambda < r$ and $f'' = \eta > D(\lambda)$, then each optimal network is an inward star;
2. if $\eta < D(\lambda)$, then each optimal network is an outward quasi-star.

Example 1 *(Optimal networks under linear link cost).* Suppose $N = \{1, 2, 3, 4\}$, $c(g_{ij}) = g_{ij}$, $b = 1$, and $\phi(z) = 1 + \lambda z$ with $\lambda > 0$.[23] When $\phi' = \lambda$ is small, the complementarity technology is weak. When λ is large, the complementarity technology is strong. Consider the four networks displayed in Fig. 5:

1. If $\lambda = 0.1$ and $f(x_i) = x_i^5$, then any G_1 with $g_{12} + g_{13} + g_{14} = 1$ is optimal;
2. If $\lambda = 0.6$ and $f(x_i) = x_i$, then G_2 is optimal;
3. If $\lambda = 0.6$ and $f(x_i) = x_i^5$, then G_3 is optimal;
4. If $\lambda = 0.6$ and $f(x_i) = -\frac{1}{x_i}$, then G_4 is optimal. ||

[23] Given $c(g_{ij}) = g_{ij}$ and $\phi(z) = 1 + \lambda z$, Assumption 1 holds if and only if $\lambda < 1$.

Finally, we discuss the relationship between the planner's optimal networks and efficient networks. Efficient networks are those that maximize the sum of agents' utilities, $\sum_i u_i(\mathbf{x}(G), G)$. When the planner's objective deviates from this goal, optimal networks may differ from efficient ones. Specifically, maximizing $\sum_i u_i(\mathbf{x}(G), G)$ results in high levels of inequality. By substituting equilibrium effort $x_i(G)$ into agents' utility functions we obtain $u_i(\mathbf{x}(G), G) = \frac{1}{2}(x_i(G))^2$. Thus, maximizing aggregate efficiency is equivalent to maximizing $\sum_i f(x_i(G)) = \sum_i (x_i(G))^2$, with $f(x_i) = x_i^2$ being strictly convex. According to Proposition 1, when f is convex and link costs are strictly convex, optimal networks tend to be strict GNSGs. Additionally, Proposition 2 states that under linear link cost, optimal networks are either inward stars or outward quasi-stars. The shared characteristic among strict GNSGs, inward stars, and a network with only two agents influencing each other is a large gap between the maximum and minimum effort among agents. In contrast, when the planner has a strictly concave f, the planner assigns greater decision weights to agents exerting less effort. Consequently, the planner constructs a network that narrows the effort gap between the best and worst agents. When f is sufficiently concave and link cost is linear, the planner creates a connected quasi-star. The planner establishes one-way links from the best agent (agent 1) to all those who would have been isolated and exerting minimum effort $x_i = \phi(0)$ in efficient networks. Although this design does not eliminate inequality, it can significantly reduce it.

4. Decentralized networks are inefficient generalized nested split graphs

We adapt the models of Bala and Goyal (2000), Galeotti and Goyal (2010), and Baetz (2015) to examine the decentralized formation of weighted and directed networks. We consider a two-stage network formation game.

Stage 1. Each agent chooses a vector $g_i = (g_{i1}, g_{i2}, \ldots, g_{i,i-1}, g_{i,i+1}, \ldots, g_{in})$, where $g_{ij} \geq 0$ is the strength of influence from agent j to agent i. Let $G \in \mathbb{R}_+^{n\times n}$ be the matrix that collects the decisions of all agents, with zeros on its main diagonal. Let $g_{-i} = (g_j)_{j\neq i}$ denote the decisions of agents other than i. We use (g_i, g_{-i}) to denote G.

Stage 2. Each agent observes G and exerts effort $x_i \geq 0$. Then, the game ends, and the payoffs for agent i are provided by

$$u_i(\mathbf{x}, G) = \phi\left(\sum_{j\neq i} g_{ij}x_j\right)x_i - \sum_{j\neq i} c(g_{ij}) - \frac{1}{2}x_i^2.$$

Assume $\phi' > 0, c'(0) = 0$ and $c'' > 0$.

In Stage 2, each agent takes G as given and chooses x_i that satisfies the best-response condition $x_i = \phi(\sum_j g_{ij}x_j)$. Let $\mathbf{x}(G)$ be the equilibrium effort profile given G. We call G an *equilibrium network* if, for each $i \in N$, we have $u_i(\mathbf{x}(g_i, g_{-i}), g_i, g_{-i}) \geq u_i(\mathbf{x}(g_i', g_{-i}), g_i', g_{-i})$ for each $g_i' \geq 0$. In other words, if G is an equilibrium network, then $(G, \mathbf{x}(G))$ constitutes a subgame perfect equilibrium of the decentralized network formation game. A network G is *(aggregate) efficient* if it maximizes $\sum_{i\in N} u_i(\mathbf{x}(G), G)$; otherwise, it is *(aggregate) inefficient*.

Let $x_i = x_i(G)$ and $z_i = \sum_j g_{ij}x_j$. By Lemma 3, the first-order condition for g_{ij} in an equilibrium network is

$$\frac{\partial u_i(\mathbf{x}(G), G)}{\partial g_{ij}} = x_i m_{ii}\phi'(z_i)x_j - c'(g_{ij}) = 0. \tag{9}$$

This condition implies that all equilibrium networks are GNSGs and inefficient.

Theorem 2. *In the decentralized network formation game, every equilibrium network is an inefficient GNSG.*

The inefficiency arises through two channels, both stemming from agents' neglect of externalities on others. First, the weight of each link in an equilibrium network is less than the efficient level. To maximize aggregate efficiency we need the following condition:

$$\frac{\partial \sum_k u_k(\mathbf{x}(G), G)}{\partial g_{ij}} = x_i m_{ii}\phi'(z_i)x_j + \underbrace{\sum_{k\neq i} x_k m_{ki}\phi'(z_i)x_j}_{\text{Externalities on other agents}} - c'(g_{ij}) = 0.$$

An increase in g_{ij} not only increases i's effort and utility but also imposes positive externalities on all those influenced directly or indirectly by i. When choosing the weight of a link, agent i ignores these positive externalities and thus chooses a link weight less than the aggregate efficient level.

Second, agents may fail to coordinate on a strictly hierarchical network that maximizes agents' total utilities with the same costs as an equilibrium network. Let G^* be an equilibrium network. Consider the planner's problem of choosing G to maximize $\sum_i u_i(\mathbf{x}(G), G)$ subject to $\sum_i \sum_j c(g_{ij}) = \sum_i \sum_j c(g_{ij}^*)$.[24] Does the equilibrium network G^* also solve this planner's problem? In general, the answer is no. Note $\phi(\sum_{j\neq i} g_{ij}x_j)x_i - \frac{1}{2}x_i^2 = \frac{1}{2}(x_i(G))^2$. Hence, the planner attempts to

[24] The constraint could be $\sum_i \sum_j c(g_{ij}) \leq \sum_i \sum_j c(g_{ij}^*)$ without changing the results, but the exposition would be lengthier.

maximize $\sum_i f(x_i(G))$ where $f(x_i) = \frac{1}{2}x_i^2$ is convex.[25] We have solved this problem. By Proposition 1, if f is convex, ϕ is convex or modestly concave, and c is modestly convex, then each optimal network is a strict GNSG in which all agents are strictly ordered by link-dominance. In contrast, in an equilibrium network, an agent need not strictly link-dominate anther. For example, a complete network with equally weighted links such as that in Fig. 3b can be an equilibrium network, but it is not optimal. This comparison demonstrates that due to each agent's neglect of externalities, agents can fail to coordinate on a strictly hierarchical network that maximizes total utilities.

5. Conclusion

To date, various network data sets have provided sufficient information to construct weighted and directed networks among the players. Examples include the peer effect data collected by Mas and Moretti (2009), global trade volume data used by König et al. (2014), and knowledge spillover data for innovation networks across various industrial fields (Acemoglu et al., 2016). However, previous studies on network design problems have predominantly focused on unweighted and undirected networks (Belhaj et al., 2016; Hiller, 2017; Corbo et al., 2006; König et al., 2014). In this paper, we address the problem of designing a weighted and directed network under complementarities. While prior research has been limited to linear agent best-responses and convex objective functions for the planner, we allow for nonlinear best-responses and arbitrarily concave objective functions.

We derive three results. First, we generalize the concept of nested split graphs previously defined for unweighted and undirected networks to weighted and directed networks by introducing the notion of link-dominance. Second, we show that every optimal weighted and directed network under complementarities is a GNSG, regardless of whether the planner's objective function is convex or strictly concave. Furthermore, in various environments, optimal GNSGs are strictly hierarchical and involve asymmetric links of varying strength, going beyond the domain of unweighted and undirected networks. Consequently, if agents' payoffs are monotone in effort, then inequality arise endogenously. Third, every equilibrium network in a non-cooperative network formation game is an inefficient GNSG. The inefficiency arises because agents choose less weighted links than the socially optimal level and because they fail to coordinate on a sufficiently hierarchical structure.

We have assumed that each agent's effort is unbounded and imposed Assumption 1 to guarantee the existence of a unique equilibrium effort profile for each feasible network. One may wonder what will happen if each agent's effort is bounded. Belhaj et al. (2014) show that bounding the effort can lead to multiple equilibria. Thus, while this case is intriguing and warrants further investigation, it cannot be easily extrapolated from our current results. Nevertheless, we offer the following conjectures. First, Belhaj et al. (2014) have derived several sufficient conditions on the best-response function ϕ to ensure that the equilibrium effort profile is unique for a given network, e.g., when ϕ is concave. If any of these conditions hold, then $\mathbf{x}(G)$ exists and is unique. Next, we observe that imposing a bound on effort would not alter the effort ordering of agents; it may at most cause some agents who initially exerted strictly more effort to now exert weakly more. Hence, link-dominance relationships would still hold, and optimal networks would continue to be GNSGs. However, Proposition 1 could change if the bound on effort is sufficiently restrictive. In that case, optimal networks could be a partially strict GNSG such that some agents exert the same maximum effort—they pool at the top.

Declaration of competing interest

This study has not been submitted for publication elsewhere and is solely the author's work. Most parts of this study were conducted during the author's Ph.D. study at the School of Economics, University of Nottingham, in the UK from 2013 to 2018, and the Economic and Social Research Council funded my Ph.D. study. There are no additional interests to declare.

Data availability

Data will be made available on request.

Appendix A. Proofs

Proof of Lemma 1. Let $\phi_i = \phi(\sum_j g_{ij}x_j)$ and $\Phi = (\phi_1, \phi_2, \ldots, \phi_n)$. That is, $\Phi(\mathbf{x}, G) \in \mathbb{R}^n$ is the best-response mapping from $(\mathbf{x}, G)$ to a (new) effort profile as defined by (3). Let Φ' be the n-by-n derivative matrix of Φ with respect to $\mathbf{x}$, the ijth element of which is $\Phi'_{ij} = \phi'(\sum_j g_{ij}x_j)g_{ij}$. Consider a feasible G. By the mean value function (e.g., Rudin, 1976, Theorem 9.19), Assumption 1 implies $\|\Phi(\mathbf{x}, G) - \Phi(\mathbf{y}, G)\| < \phi'\|G\||\mathbf{x} - \mathbf{y}| \le d|\mathbf{x} - \mathbf{y}|$ for $d < 1$. Thus, $\Phi(\cdot, G)$ is a contraction. Hence, according to the Banach fixed point theorem (Ok, 2011, Ch. C.6), a unique fixed point $\mathbf{x}(G)$ exists such that $\Phi(\mathbf{x}, G) = \mathbf{x}$. By the assumption $\phi(0) = 1 > 0$, we further have $\mathbf{x}(G) \in \mathbb{R}^n_+$. Next, observe that the spectral radius of Φ', $\rho(\Phi')$, satisfies $\rho(\Phi') \le \phi'\|G\| < 1$ for each feasible network G. Thus, by Debreu and Herstein (1953), $A = [\Phi' - I]$ is invertible. It then

[25] Since $\sum_i \sum_j c(g_{ij}) = \sum_i \sum_j c(g^*_{ij})$ is a constant in the planner's problem, maximizing $\sum_i u_i(\mathbf{x}, G)$ is equivalent to maximizing $\sum_i [\phi(\sum_{j \neq i} g_{ij}x_j)x_i - \frac{1}{2}x_i^2]$ plus a constant, which results in maximizing $\sum_i \frac{1}{2}(x_i(G))^2$.

follows from the implicit function theorem (Rudin, 1976, Theorem 9.28) that $\mathbf{x}(G)$ is continuously differentiable at each feasible G.

Proof of Lemma 2. By Lemma 1, $x_i(G)$ is continuous for each feasible G. Thus, $\pi(G) = \sum_i f(x_i(G))$ is continuous for each feasible G. Also, the set of feasible networks is a closed, bounded subset of $\mathbb{R}^{n\times n}$ and thus, compact. Hence, by Berge's maximum theorem (Ok, 2011, p. 306), the planner's problem has a solution.

Proof of Lemma 3. Given the matrix M, we have $\frac{\partial x_k(G)}{\partial g_{ij}} = \sum_{\ell\in N} m_{k\ell}\frac{\partial \phi(z_\ell)}{\partial g_{ij}} = m_{ki}\phi'(z_i)x_j(G)$, where the first equality follows from the implicit function theorem for vector-valued functions, and the second equality follows from $\frac{\partial \phi(z_i)}{\partial g_{ij}} = \phi'(z_i)x_j$ and the fact that $\frac{\partial \phi(z_\ell)}{\partial g_{ij}} = 0$ for each $\ell \neq i$. A version of the implicit function theorem can be found in Rudin (1976, Theorem 9.28) which implies $\mathbf{x}'(G) = -[\Phi' - I]^{-1}\frac{\partial \Phi}{\partial G} = M\frac{\partial \Phi}{\partial G}$, where $\mathbf{x}'(G)$ is the n-by-n^2 matrix whose (k,ij)-th element is $\frac{\partial x_k(G)}{\partial g_{ij}}$ and $\frac{\partial \Phi}{\partial G}$ is the n-by-n^2 matrix whose (k,ij)-th element is $\frac{\partial \phi_k}{\partial g_{ij}}$. Substituting $\frac{\partial x_k(G)}{\partial g_{ij}} = m_{ki}\phi'(z_i)x_j(G)$ into (5), we obtain $MV_{ij}(G) = \alpha_i(G)x_j(G)$.

Proof of Lemma 4. First, we prove the formula for α_i that we state in the main text.

Lemma 5. *Let G be a feasible network, $\alpha_i = \alpha_i(G)$, $x_i = x_i(G)$ and $z_i = \sum_j g_{ij}x_j$. Then, $\alpha_i = f'(x_i)\phi'(z_i) + \sum_{j\neq i}\alpha_j g_{ji}\phi'(z_i)$.*

Proof of Lemma 5. Given Assumption 1, we have $\sum_{l=0}^{\infty}(\Phi')^l = [I-\Phi']^{-1} = M$. Hence $M\Phi' = \left(\sum_{l=0}^{\infty}(\Phi')^l\right)\Phi' = \sum_{l=1}^{\infty}(\Phi')^l = M - I$. Therefore, $M = M\Phi' + I$. Thus, $m_{ii} = \sum_k m_{ik}\phi_k' g_{ki} + 1$, and for each $i, j \in N$, $i \neq j$, $m_{ij} = \sum_k m_{ik}\Phi'_{kj} = \sum_k m_{ik}\phi'(z_k)g_{kj}$. It follows that

$$
\begin{aligned}
\alpha_i &= f'(x_i)m_{ii}\phi'(z_i) + \sum_{k\neq i} f'(x_k)m_{ki}\phi'(z_i) \\
&= f'(x_i)\phi'(z_i) + \sum_k f'(x_k)\Big(\sum_{j\neq i} m_{kj}\phi_j' g_{ji}\Big)\phi'(z_i) \\
&= f'(x_i)\phi'(z_i) + \sum_{j\neq i}\Big(\sum_k f'(x_k)m_{kj}\phi_j'\Big)g_{ji}\phi'(z_i) \\
&= f'(x_i)\phi'(z_i) + \sum_{j\neq i}\alpha_j g_{ji}\phi'(z_i). \quad \square
\end{aligned}
$$

In the following we assume $c'' > 0$. Let μ be the Lagrangian multiplier for the planner's optimization problem. We show the first two statements of Lemma 4; the third statement follows immediately from the first two.

First, suppose $\alpha_i \geq \alpha_j$. To show $g_{ik} \geq g_{jk}$ for each $k \neq i, j$, take an agent $k \in N, k \neq i, j$. Then, $MV_{ik} = \alpha_i x_k \geq \alpha_j x_k = MV_{jk}$. If $g_{jk} = 0$, then trivially $g_{ik} \geq g_{jk}$. If $g_{jk} > 0$, then by the KT (Kuhn-Tucker) condition, $\mu c'(g_{ik}) \geq \alpha_i x_k \geq \alpha_j x_k = \mu c'(g_{jk})$, again implying $g_{ik} \geq g_{jk}$ given $c'' > 0$. To show $x_i(G) \geq x_j(G)$, suppose that the opposite is true. Then, $\sum_k g_{ik}x_k < \sum_k g_{jk}x_k$. Since $g_{ik} \geq g_{jk}$ for each $k \neq i, j$, we have $g_{ij}x_j < g_{ji}x_i$; thus, $g_{ij} < g_{ji}$. However, according to the KT condition and $\alpha_j x_i < \alpha_i x_j$, we have $\mu c'(g_{ji}) \leq \mu c'(g_{ij})$, implying $g_{ij} \geq g_{ji}$—a contradiction. Hence, $x_i(G) \geq x_j(G)$.

Second, suppose $x_i \geq x_j$. To show $g_{ki} \geq g_{kj}$ for each $k \neq i, j$, take an agent $k \in N, k \neq i, j$. Then, $MV_{ki} = \alpha_k x_i \geq \alpha_k x_j = MV_{kj}$. If $g_{kj} = 0$, then trivially $g_{ki} \geq g_{kj}$. If $g_{kj} > 0$, then by the KT condition, $\mu c'(g_{ki}) \geq \mu c'(g_{kj})$. This implies $g_{ki} \geq g_{kj}$ given $c'' > 0$. Next, to show that $\alpha_i \geq \alpha_j$, we suppose $\alpha_i < \alpha_j$ for contradiction. Then, $g_{ik} \leq g_{jk}$ for each $k \neq i, j$, and $x_i \leq x_j$. Hence, $x_i = x_j$, and $g_{ki} = g_{kj}$ for each $k \neq i, j$. Therefore, by applying Lemma 5, we obtain $\alpha_i - \alpha_j = (\alpha_j g_{ji} - \alpha_i g_{ij})\phi'(z_i)$. Hence, $\alpha_i < \alpha_j$ implies $\alpha_j g_{ji} < \alpha_i g_{ij}$, leading to $g_{ij} > g_{ji}$. However, by the suppositions $x_i \geq x_j$ and $\alpha_i < \alpha_j$, we have $MV_{ij} = \alpha_i x_j < \alpha_j x_i = MV_{ji}$, implying $g_{ij} \leq g_{ji}$ – a contradiction. Hence, we have $\alpha_i \geq \alpha_j$.

Proof of Proposition 1. We consider a feasible network G with $g_{ij} > 0$ for each $i \neq j$ and suppose for contradiction that G is optimal and $x_i(G) = x_j(G)$ for some $i \neq j$. To simplify notations and without loss of generality, we assume $x_1(G) = x_2(G)$ (the indexes need not reflect the ranking of efforts). Then we show that there is an alternative feasible network G^ϵ such that $\pi(G^\epsilon) > \pi(G)$ – hence a contradiction. Let $\epsilon > 0$. The network $G^\epsilon = (g_{ij}^\epsilon)$ is obtained from $G = (g_{ij})$ by reallocating weights:

$$
\begin{aligned}
&g_{31}^\epsilon = g_{31} + \epsilon, \quad g_{32}^\epsilon = c^{-1}(c(g_{31}) + c(g_{32}) - c(g_{31}^\epsilon)), \\
&g_{13}^\epsilon = g_{13} + \epsilon, \quad g_{23}^\epsilon = c^{-1}(c(g_{13}) + c(g_{23}) - c(g_{13}^\epsilon));
\end{aligned}
$$

we let $g^\epsilon_{ij} = g_{ij}$ for all other elements in G^ϵ. The variable $\epsilon > 0$ is chosen sufficiently small so that g^ϵ_{32} and g^ϵ_{23} are strictly positive. By construction, $\sum_i \sum_j c(g^\epsilon_{ij}) = \sum_i \sum_j c(g_{ij}) \le b$. The proof has four steps. Steps 1 and 2 establish some basic facts. Steps 3 and 4 examine the derivatives $\pi' \equiv \frac{\partial \pi(G^\epsilon)}{\partial \epsilon}$ and $\pi'' \equiv \frac{\partial \pi'}{\partial \epsilon}$ respectively and show that if h and $\phi 6$ are not too concave and c not too convex, then $\pi' = 0$ and $\pi'' > 0$ at $\epsilon = 0$. It then follows from the Taylor's theorem (Rudin, 1976, Theorem 5.15) that $\pi(G^\epsilon) > \pi(G)$ for a sufficiently small $\epsilon > 0$. Let $\mathcal{F}$ be the set of feasible networks.

Step 1. We show $\alpha_1(G) = \alpha_2(G)$, $g_{j1} = g_{j2}$ and $g_{1j} = g_{2j}$ for each $j \ge 3$, and $g_{12} = g_{21}$. Lemma 4 directly implies $g_{j1} = g_{j2}$ and $g_{1j} = g_{2j}$ for each $j \ge 3$, and $\alpha_1 = \alpha_2$. To show $g_{12} = g_{21}$, we apply Lemma 5 and obtain $\alpha_1 - \alpha_2 = \alpha_2 g_{21}\phi'(z_1) - \alpha_1 g_{12}\phi'(z_2) = (g_{21} - g_{12})\alpha_2\phi'(z_1) = 0$. Hence, $g_{21} = g_{12}$.

Step 2. Given $c(g^\epsilon_{32}) + c(g^\epsilon_{31}) = c(g_{31}) + c(g_{32})$, taking derivatives w.r.t. ϵ we obtain $c'(g^\epsilon_{32})\frac{\partial g^\epsilon_{32}}{\partial \epsilon} + c'(g^\epsilon_{31}) = 0$. Thus,

$$\frac{\partial g^\epsilon_{32}}{\partial \epsilon} = -\frac{c'(g^\epsilon_{31})}{c'(g^\epsilon_{32})}, \quad \text{implying} \quad \frac{\partial g^\epsilon_{32}}{\partial \epsilon}\Big|_{\epsilon=0} = -\frac{c'(g_{31})}{c'(g_{31})} = -1.$$

Further, $c''(g^\epsilon_{32})(\frac{\partial g^\epsilon_{32}}{\partial \epsilon})^2 + c'(g^\epsilon_{32})\frac{\partial g^\epsilon_{32}}{\partial \epsilon} + c''(g^\epsilon_{31}) = 0$. Hence, $\frac{\partial^2 g^\epsilon_{32}}{\partial \epsilon^2}|_{\epsilon=0} = -\frac{2c''(g_{31})}{c'(g_{31})}$. Let μ be the Lagrangian multiplier w.r.t. G. Using the fact $\mu = \frac{\alpha_i(G)x_j(G)}{c'(g_{ij})}$ for all networks satisfying the KT first-order conditions we obtain an upper bound of μ: $\bar{\mu} = \frac{\alpha^H x^H}{c'\left(c^{-1}\left(\frac{b}{n(n-1)}\right)\right)}$, where $\alpha^H = \max_{G\in\mathcal{F}, i\in N} \alpha_i(G)$ and $x^H = \max_{G\in\mathcal{F}, i\in N} x_i(G)$. Hence, $c'(g_{31}) = \frac{\alpha_3(G)x_1(G)}{\mu} \ge \alpha^L x^L/\bar{\mu}$, where $\alpha^L = \min_{G\in\mathcal{F}, i\in N} \alpha_i(G) > 0$ and $x^L = \min_{G\in\mathcal{F}, i\in N} x_i(G) > 0$. Since $\alpha_i(G)$ and $x_i(G)$ are continuous in G and $\mathcal{F}$ is compact, the maximums and the minimums exist. It follows that, by $c'' \le t$,

$$\frac{\partial^2 g^\epsilon_{32}}{\partial \epsilon^2}\Big|_{\epsilon=0} = -\frac{2c''(g_{31})}{c'(g_{31})} \ge -\frac{2\bar{\mu}}{\alpha^L x^L}t.$$

Similarly, by $g^\epsilon_{13} + g^\epsilon_{23} = c(g_{13}) + c(g_{23})$, we obtain $\frac{\partial g^\epsilon_{23}}{\partial \epsilon} = -\frac{c'(g^\epsilon_{13})}{c'(g^\epsilon_{23})}$, implying $\frac{\partial g^\epsilon_{23}}{\partial \epsilon}|_{\epsilon=0} = -\frac{c'(g_{13})}{c'(g_{13})} = -1$ and $\frac{\partial^2 g^\epsilon_{23}}{\partial \epsilon^2}|_{\epsilon=0} = -\frac{2c''(g_{13})}{c'(g_{13})} \ge -\frac{2\bar{\mu}}{\alpha^L x^L}t$. Hence, by choosing a sufficiently small t, $\frac{\partial^2 g^\epsilon_{32}}{\partial \epsilon^2}|_{\epsilon=0}$ and $\frac{\partial^2 g^\epsilon_{23}}{\partial \epsilon^2}|_{\epsilon=0}$ can be arbitrarily chose to zero.

Step 3. We show $\pi'|_{\epsilon=0} = 0$. Let $x_i = x_i(G^\epsilon)$ and $\alpha_i = \alpha_i(G^\epsilon)$. Since $MV_{ij}(G^\epsilon) = \alpha_i x_j$ and $g^\epsilon_{ij} = g_{ij}$ for all links unrelated to agents 1, 2 and 3, we have

$$\pi' = \sum_i \sum_j \alpha_i x_j \frac{\partial g^\epsilon_{ij}}{\partial \epsilon} = \alpha_1 x_3 + \alpha_2 x_3 \frac{\partial g^\epsilon_{23}}{\partial \epsilon} + \alpha_3 x_1 + \alpha_3 x_2 \frac{\partial g^\epsilon_{32}}{\partial \epsilon}.$$

At $\epsilon = 0$, we have $x_1 = x_2$, $\alpha_1 = \alpha_2$, and $\frac{\partial g^\epsilon_{23}}{\partial \epsilon} = \frac{\partial g^\epsilon_{32}}{\partial \epsilon} = -1$. Hence, $\pi'|_{\epsilon=0} = 0$.

Step 4. We show that there are lower bounds for h'' and ϕ'' to ensure $\pi''|_{\epsilon=0} > 0$. Let $A \equiv \alpha_1 + \alpha_2 \frac{\partial g^\epsilon_{23}}{\partial \epsilon}$, $B \equiv x_1 + x_2 \frac{\partial g^\epsilon_{32}}{\partial \epsilon}$, $\alpha'_i \equiv \frac{\partial \alpha_i(G^\epsilon)}{\partial \epsilon}$, and $x'_i \equiv \frac{\partial x_i(G^\epsilon)}{\partial \epsilon}$. Then $\pi' = Ax_3 + B\alpha_3$, and $A|_{\epsilon=0} = B|_{\epsilon=0} = 0$. Thus, $\pi''|_{\epsilon=0} = A'x_3 + B'\alpha_3$. Let $\Delta_\alpha \equiv \left(\alpha'_1 - \alpha'_2\right)|_{\epsilon=0}$ and $\Delta_x \equiv \left(x'_1 - x'_2\right)|_{\epsilon=0}$. Then,

$$\begin{aligned}\pi''|_{\epsilon=0} &= (\Delta_\alpha + \alpha_2 \frac{\partial^2 g^\epsilon_{23}}{\partial \epsilon^2})x_3 + (\Delta_x + x_2 \frac{\partial^2 g^\epsilon_{32}}{\partial \epsilon^2})\alpha_3 \\ &\ge \Delta_\alpha x_3 + \Delta_x \alpha_3 - t\frac{2\bar{\mu}}{\alpha^L x^L}(\alpha_2 x_3 + \alpha_3 x_2).\end{aligned}$$

At $\epsilon = 0$, we have

$$x'_1 = \phi'(z_1)\Big(\sum_{j\ge 4} g_{1j}x'_j + g_{13}x_3 + g_{13}x'_3 + g_{12}x'_2\Big)$$

$$x'_2 = \phi'(z_2)\Big(\sum_{j\ge 4} g_{2j}x'_j - g_{23}x_3 + g_{23}x'_3 + g_{21}x'_1\Big).$$

By Step 1, $z_1 = z_2$, $g_{j1} = g_{j2}$ and $g_{1j} = g_{2j}$ for each $j \ge 3$, and $g_{12} = g_{21}$. Thus, we obtain

$$\Delta_x = \left(x'_1 - x'_2\right)|_{\epsilon=0} = \frac{2\phi'(z_1)x_3}{1 + \phi'(z_1)g_{12}} > 0.$$

Because Δ_x is a combination of continuous functions of G and $\mathcal{F}$ is compact, Δ_x is bounded below and away from zero. Denote $\Delta^L_x \equiv \inf_{G\in\mathcal{F}} \Delta_x$. Further, let $z'_i \equiv \frac{\partial z_i(G^\epsilon)}{\partial \epsilon}$. Since $x'_i = \phi'(z_i)z'_i$, we have $\Delta_z \equiv \left(z'_1 - z'_2\right)|_{\epsilon=0} = \frac{2x_3}{1+\phi'(z_1)g_{12}}$, which is positive and bounded above.

Finally, let $h(z)=f(\phi(z))$. By Lemma 5, at $\epsilon=0$,

$$\alpha_1' = h''(z_1)z_1' + \phi'(z_1)\Big(\sum_{j\geq 4}\alpha_j' g_{j1} + \alpha_3' g_{31} + \alpha_3 + \alpha_2' g_{21}\Big) + \sum_j \alpha_j g_{j1}\phi''(z_1)z_1'$$
$$\alpha_2' = h''(z_2)z_2' + \phi'(z_2)\Big(\sum_{j\geq 4}\alpha_j' g_{j2} + \alpha_3' g_{32} - \alpha_3 + \alpha_1' g_{12}\Big) + \sum_j \alpha_j g_{j2}\phi''(z_2)z_2'.$$

Thus,

$$\Delta_\alpha = \left(\alpha_1' - \alpha_2'\right)|_{\epsilon=0} = \frac{[h''(z_1)+\phi''(z_1)\sum_j \alpha_j g_{j1}]\Delta_z + 2\alpha_3\phi'(z_1)}{1+\phi'(z_1)g_{12}}.$$

If $h''(z_1)+\phi''(z_1)\sum_j\alpha_j g_{j1}\geq 0$, then $\Delta_\alpha>0$, and thus by choosing a sufficiently small $t>0$, we obtain

$$\pi''|_{\epsilon=0} > \Delta_x\alpha_3 - t\frac{2\bar{\mu}}{\alpha^L x^L}(\alpha_2 x_3+\alpha_3 x_2) \geq \frac{1}{2}\Delta_x^L\alpha^L > 0.$$

Suppose $h''(z_1)+\phi''(z_1)\sum_j\alpha_j g_{j1}<0$ but $h''\geq -r$ and $\phi''\geq -r$ for some $r>0$. Then, $[1+\phi'(z_1)g_{12}]\Delta_\alpha \geq -r(1+\sum_j\alpha_j g_{j1})\Delta_z + 2\alpha_3\phi'(z_1)$. Since Δ_z is bounded above and $\alpha_3\phi'(z_1)$ is bounded below and away from zero, we again have $\Delta_\alpha>0$ by choosing a sufficiently small r. Hence, there are sufficiently small thresholds $r>0$, $s=r$ and $t>0$ such that if $h''\geq -r$, $\phi''\geq -s$ and $c''\leq t$ then $\pi''|_{\epsilon=0}>\frac{1}{2}\Delta_x^L\alpha^L>0$ even if h'' and ϕ'' are negative.

Step 5. We have shown that if $x_i(G)=x_j(G)$ then $\pi'|_\epsilon=0$ and $\pi''|_\epsilon>0$ so that there is $\epsilon>0$ such that $\pi(G^\epsilon)>\pi(G)$. Now suppose $x_i(G)>x_j(G)$. Then $MV_{ki}=\alpha_k x_i>\alpha_k x_j=MV_{kj}$, and thus $g_{ki}>g_{kj}$, for each $k\neq i,j$. Also, by Lemma 4, $x_i>x_j$ implies $\alpha_i>\alpha_j$. Thus, for each $k\neq i,j$, we have $MV_{ik}=\alpha_i x_k>\alpha_j x_k=MV_{jk}$, and thus $g_{ki}>g_{kj}$. Hence, $i\succ_G j$. This completes the proof of Proposition 1.

Proof of Proposition 2. Consider a feasible network G and suppose $x_{j_1}(G)\geq x_{j_2}(G)\geq\cdots\geq x_{j_n}(G)$. We say that $\hat{G}$ first-order stochastically dominates (FSD) G if for each $i\in N$ and each $\bar{k}\in N\backslash\{i\}$, $\sum_{k=1}^{\bar{k}}\hat{g}_{i,j_k}\geq\sum_{k=1}^{\bar{k}}g_{i,j_k}$. That is, given the status quo effort profile $x(G)$, if each agent in network $\hat{G}$ puts greater weights on inward links leading from neighbors with greater efforts, then $\hat{G}$ FSD G. We write $\hat{\mathbf{x}}\geq\mathbf{x}$ if $\hat{x}_i\geq x_i$ for each $i\in N$.

Lemma 6. *If G and $\hat{G}$ are both feasible, and $\hat{G}$ FSD G, then $\mathbf{x}(\hat{G})\geq\mathbf{x}(G)$.*

Proof. Since $x(G)$ exists and is unique for each feasible G given Assumption 1, as a fixed point to the best-response mapping $\Phi(\cdot,G)$ we obtain $\mathbf{x}(G)$ by $\mathbf{x}(G)=\sup\{\mathbf{x}|\Phi(\mathbf{x},G)\geq x\}$. Let $\mathbf{x}=\mathbf{x}(G)$. Then, given $\hat{G}$ FSD G, we have for each $i\in N$:

$$\phi(\sum_{j\in N}\hat{g}_{ij}x_j)=\phi(\sum_{k\in N}\hat{g}_{i,j_k}x_{j_k})\geq\phi(\sum_{k\in N}g_{i,j_k}x_{j_k})=\phi(\sum_{j\in N}g_{i,j}x_j).$$

Hence, $\Phi(\mathbf{x},\hat{G})\geq\Phi(\mathbf{x},G)=\mathbf{x}(G)$. Therefore, $\mathbf{x}(G)\in\{\mathbf{x}|\Phi(\mathbf{x},\hat{G})\geq\mathbf{x}\}$. Thus, $\mathbf{x}(\hat{G})=\sup\{\mathbf{x}|\Phi(\mathbf{x},\hat{G})\geq\mathbf{x}\}\geq\mathbf{x}(G)$. □

In the remaining steps, suppose that G is optimal and that G is not an inward star or an outward quasi-star. Without loss of generality, assume $x_1(G)\geq x_2(G)\geq\cdots\geq x_n(G)$. Consider $\hat{G}$ obtained from G such that 1) $\hat{g}_{12}=\sum_{j\neq 1}g_{1j}$, 2) $\hat{g}_{i1}=\sum_{j\neq i}g_{ij}$ for each $i\in N, i\geq 2$, and 3) $\hat{g}_{1j}=0$ for each $j\neq 2$ and $\hat{g}_{ij}=0$ for each $i\geq 2, j\neq 1$. By such construction we have $\sum_j\hat{g}_{ij}=\sum_j g_{ij}$ for each $i\in N$ so that $\sum_i\sum_j c(\hat{g}_{ij})=\sum_i\sum_j c(g_{ij})$.

Step 1. For $i=1$, observe $\sum_{j=2}^k\hat{g}_{1j}=\hat{g}_{12}=\sum_{j\geq 2}g_{1j}\geq\sum_{j=2}^k g_{1j}$ for each $k\geq 2$. For each $i\geq 2$, observe $\sum_{j=1}^k\hat{g}_{ij}=\hat{g}_{i1}=\sum_j g_{ij}\geq\sum_{j=1}^k g_{ij}$ for each $k\neq i$. Hence, $\hat{G}$ FSD G.

By Lemma 6 it follows that $\mathbf{x}(\hat{G})\geq\mathbf{x}(G)$. Since G is already optimal, we obtain $\mathbf{x}(\hat{G})=\mathbf{x}(G)$, and that $\hat{G}$ is also optimal.

Step 2. That G is not an inward star or an outward quasi-star implies: 1) $g_{ik}>0$ for some $i\in N, k\geq 3$, or 2) $g_{k2}>0$ for some $k\geq 3$, or both. In either case, we show that there are contradictions. In the following, let $x_i=x_i(G)$ for each $i\in N$.

Case 1: $g_{ik}>0$ for some $i\in N, k\geq 3$. Without loss of generality, assume $g_{i3}>0$ for some i. If $g_{13}>0$, then by $x_1(\hat{G})=x_1(G)$ we have $\phi(\hat{g}_{12}x_2)=\phi(\sum_{j\neq 3}g_{1j}x_2+g_{13}x_2)=\phi(\sum_{j\neq 3}g_{1j}x_j+g_{13}x_3)$. But $x_2\geq x_j$ for each $j\geq 3$. Hence, $x_2=x_3$. If $g_{i3}>0$ for some $i\geq 2$, then $\phi(\hat{g}_{i1}x_1)=\phi(\sum_{j\neq 3}g_{ij}x_1+g_{i3}x_1)=\phi(\sum_{j\neq 3}g_{ij}x_j+g_{i3}x_3)$. Hence, $x_1=x_3$. Since $x_1\geq x_2\geq x_3$, we again have $x_2=x_3$.

However, by $x_2=x_3$, we have $\phi(\hat{g}_{21}x_1)=\phi(\hat{g}_{31}x_1)$, implying $\hat{g}_{21}=\hat{g}_{31}$. First, if $\hat{g}_{21}=\hat{g}_{31}>0$, then $MV_{21}(\hat{G})=MV_{31}(\hat{G})$, leading to $\alpha_2(\hat{G})x_1=\alpha_3(\hat{G})x_1$. Hence, $\alpha_2(\hat{G})=\alpha_3(\hat{G})$. Then, by Lemma 5, a contradiction follows: in network $\hat{G}$,

$$\alpha_2(\hat{G})=f'(x_2)\phi'(z_2)+\alpha_1(\hat{G})\hat{g}_{12}\phi'(z_2)>f'(x_3)\phi'(z_3)=\alpha_3(\hat{G}).$$

Second, if $\hat{g}_{21} = \hat{g}_{31} = 0$, then since $x_2 \geq x_i$ for each $i \geq 2$ we have $\hat{g}_{i1} = 0$ for each $i \geq 2$. Hence, $g_{ij} = 0$ for each $i \geq 2$, $j \in N$, so that G is an inward star – contradicting our suppositions.

Case 2: $g_{k2} > 0$ for some $k \geq 3$. Then, by $x_k(\hat{G}) = x_k(G)$, we have $\phi(\sum_j g_{kj}x_1) = \phi(\sum_j g_{kj}x_j)$, implying $x_1 = x_2$. Hence, $\hat{g}_{12} = \hat{g}_{21}$. Network G is optimal. Thus, it is not empty. Therefore, $\hat{g}_{12} = \hat{g}_{21} > 0$. It follows that $\alpha_1(\hat{G})x_2 = \alpha_2(\hat{G})x_1$, implying $\alpha_1(\hat{G}) = \alpha_2(\hat{G})$. However,

$$\alpha_1(\hat{G}) \geq f'(x_1)\phi'(z_1) + \alpha_2(\hat{G})\hat{g}_{21}\phi'(z_1) + \alpha_k(\hat{G})\hat{g}_{k1}\phi'(z_1),$$
$$\alpha_2(\hat{G}) = f'(x_2)\phi'(z_2) + \alpha_1(\hat{G})\hat{g}_{12}\phi'(z_2),$$

and thus,

$$\alpha_1(\hat{G}) - \alpha_2(\hat{G}) \geq \frac{\alpha_k(\hat{G})\hat{g}_{k1}\phi'(z_1)}{1 + \hat{g}_{21}\phi'(z_1)} > 0$$

since $\hat{g}_{k1} \geq g_{k2} > 0$. Hence, we obtain a contradiction, completing our proof of Proposition 2.

Proof of Remark 2. First, suppose $f'' = \eta > 0$. Let G be optimal. Suppose G is an outward quasi-star, with $x_1(G) \geq x_2(G) > x_i(G)$ for each $i \geq 3$. Then, by the convexity of f, we immediately have $g_{i1} = 0$ for each $i \geq 3$; otherwise we can reallocate the weight from g_{i1} to g_{21} to achieve a strict improvement (agents 1 and 2 will strictly improve and the improvement is strictly greater than any $i \geq 3$'s reduction). Hence, the links between 1 and 2 exhaust all resources: $g_{12} + g_{21} = 1$. Let $\alpha_i = \alpha_i(G)$ and $x_i = x_i(G)$. Then, given $x_2 \geq \phi(0) = 1$,

$$\begin{aligned} MV_{12} - MV_{21} &= \alpha_1 x_2 - \alpha_2 x_1 = (\alpha_1 - \alpha_2)x_2 - (x_1 - x_2)\alpha_2 \\ &\geq \alpha_1 - \alpha_2 - (x_1 - x_2)\alpha_2. \end{aligned}$$

Since $\alpha_1 = \lambda f'(x_1) + \lambda\alpha_2 g_{21}$ and $\alpha_2 = \lambda f'(x_2) + \lambda\alpha_1 g_{12}$, we obtain $\alpha_1 \leq \lambda\frac{f'(x_1)}{1-\lambda g_{21}}$, $\alpha_2 \leq \lambda f'(x_2) + \lambda^2\frac{f'(x_1)}{1-\lambda g_{21}}$, and

$$\begin{aligned} \alpha_1 - \alpha_2 &= (1 - \lambda g_{12})\alpha_1 - \lambda f'(x_2) \\ &\geq \lambda\left[(1 - \lambda g_{12}) f'(x_1) - f'(x_2)\right]. \end{aligned}$$

Also, by $x_1 = 1 + \lambda x_2$ and $x_2 = 1 + \lambda x_1$ we obtain $x_1 - x_2 = \lambda\frac{g_{12}-g_{21}}{1-\lambda^2 g_{12}g_{21}}$. By the Taylor's theorem, we have $f'(x_1) = f'(x_2) + \eta(x_1 - x_2)$. Hence,

$$\begin{aligned} &(MV_{12} - MV_{21})/\lambda^2 \\ \geq& \frac{1}{\lambda}(1 - \lambda g_{12})\left[f'(x_2) + \eta(x_1 - x_2)\right] - \frac{1}{\lambda}f'(x_2) - \frac{1}{\lambda}(x_1 - x_2)\left(f'(x_2) + \lambda\frac{f'(x_1)}{1 - \lambda g_{21}}\right) \\ =& - g_{12}f'(x_2) + (1 - \lambda g_{12})\,\eta\frac{g_{12} - g_{21}}{1 - \lambda^2 g_{12}g_{21}} - \frac{g_{12} - g_{21}}{1 - \lambda^2 g_{12}g_{21}}\left(f'(x_2) + \lambda\frac{f'(x_1)}{1 - \lambda g_{21}}\right) \\ \equiv& \Delta(\lambda). \end{aligned}$$

Since $\lim_{\lambda\to 0} x_2 = 1$ and $g_{21} = 1 - g_{12}$, we have

$$\lim_{\lambda\to 0}\Delta(\lambda) = \eta(g_{12} - g_{21}) - (2g_{12} - g_{21})\,f'(1) = \eta(2g_{12} - 1) - (3g_{12} - 1)\,f'(1).$$

Thus, for each $g_{12} > \frac{1}{2}$, we have $\lim_{\lambda\to 0}\Delta(\lambda) > 0$ for each sufficiently large η. Moreover, if $\eta > \frac{3}{2}f'(1)$ and $\lim_{\lambda\to 0}\Delta(\lambda) \geq 0$ for some g_{12}, then $\lim_{\lambda\to 0}\Delta(\lambda) > 0$ for each network G' with $g'_{12} > g_{12}$, which implies that $\lim_{\lambda\to 0}\Delta(\lambda)$ is convex in g_{12}. Thus, there are sufficiently small λ and sufficiently large η such that $MV_{12} - MV_{21} > 0$, and $\max\pi(G)$ is achieved at either $g_{12} = g_{21} = \frac{1}{2}$ or $g_{12} = 1$.

Next, let G denote the network with $g_{12} = g_{21} = \frac{1}{2}$ and $\hat{G}$ denote the network with $\hat{g}_{12} = 1$. Let $x_i = x_i(G)$ and $\hat{x}_i = \hat{x}_i(G)$. Then, $x_1 = x_2 = \frac{1}{1-\lambda/2}$, $\hat{x}_1 = 1 + \lambda$, and $\hat{x}_2 = 1$. Hence,

$$\begin{aligned} \pi(\hat{G}) - \pi(G) &= [f(\hat{x}_1) - f(x_1)] + [f(1) - f(x_2)] \\ &= f'(x_1)(\hat{x}_1 - x_1) + \frac{1}{2}f''(x_1)(\hat{x}_1 - x_1)^2 + f'(x_1)(1 - x_1) + \frac{1}{2}f''(x_1)(1 - x_1)^2 \\ &= \frac{1}{2}\eta\left[(\hat{x}_1 - x_1)^2 + (1 - x_1)^2\right] - f'(x_1)\left(2x_1 - \hat{x}_1 - 1\right) \\ &= \frac{1}{2}\eta\lambda^2\left[1 - \lambda + \left(\frac{1}{2 - \lambda}\right)^2\right] - f'\left(\frac{1}{1 - \lambda/2}\right)\lambda^2\frac{1}{2 - \lambda}. \end{aligned} \tag{10}$$

Simplifying, we obtain $\pi(\hat{G}) - \pi(G) > 0$ if and only if

$$\eta > D(\lambda) \equiv \frac{2f'\left(\frac{2}{2-\lambda}\right)}{(2-\lambda)(1-\lambda) - 1/(2-\lambda)}. \tag{11}$$

We have $D(\lambda)$ strictly increasing in λ, since $f'' > 0$ and taking the derivative of the denominator w.r.t. λ results in $2\lambda - 3 + \frac{1}{(2-\lambda)^2} < 0$ given $\lambda < 1$. It follows that there are $r > 0$ and $s > 0$ such that if $\lambda < r$ and $\eta > s$, then the inward star $\hat{G}$ outperforms any outward quasi-star G.

Finally, observe from equation (10) and inequality (11) that it is sufficient to have $\eta < D(\lambda)$ to ensure that $\pi(\hat{G}) < \pi(G)$ and inward stars are outperformed by the outward quasi-star with $g_{12} = g_{21} = \frac{1}{2}$. This completes our proof.

Proof of Theorem 2. Step 1: Since $\frac{\partial u_i(\mathbf{x},G)}{\partial x_i} = \phi\left(\sum_{j\in N} g_{ij}x_j\right) - x_i$, $\mathbf{x}(G)$ satisfies the first-order condition: $x_i = \phi(\sum_j g_{ij}x_j)$ for each $i \in N$. Hence, $\frac{\partial x_k(G)}{\partial g_{ij}} = m_{ki}\phi'(z_i)x_j$, where $z_i = \sum_{j\in N} g_{ij}x_j$. Thus,

$$\frac{du_i(\mathbf{x}(G),G)}{dg_{ij}} = x_i\frac{\partial x_i(G)}{\partial g_{ij}} - c'(g_{ij}) = x_i m_{ii}\phi'(z_i)x_j - c'(g_{ij}).$$

It follows that if G is an equilibrium network, then G satisfies

$$\begin{cases} x_i m_{ii}\phi'(z_i)x_j \le c'(g_{ij}) & \text{for each } i,j \in N, j \ne i; \\ x_i m_{ii}\phi'(z_i)x_j = c'(g_{ij}) & \text{if } g_{ij} > 0. \end{cases} \tag{12}$$

Step 2: We show that every equilibrium network is a GNSG. Let G be an equilibrium network, and consider $i \ne j$ such that $x_i \ge x_j$. Then, for each $k \ne i, j$, $x_k m_{kk}\phi'(z_k)x_i \ge x_k m_{kk}\phi'(z_k)x_j$. If $g_{kj} = 0$, then trivially $g_{ki} \ge g_{kj}$. If $g_{kj} > 0$, then $c'(g_{ki}) \ge x_k m_{kk}\phi'(z_k)x_i \ge x_k m_{kk}\phi'(z_k)x_j = c'(g_{kj})$; thus $g_{ki} \ge g_{kj}$. In either case, $g_{ki} \ge g_{kj}$.

Next, suppose that $g_{ik} < g_{jk}$ for some $k \ne i, j$. Then $x_i m_{ii}\phi'(z_i)x_k \le x_j m_{jj}\phi'(z_j)x_k$. Thus $x_i m_{ii}\phi'(z_i) \le x_j m_{jj}\phi'(z_j)$. Therefore, $x_i m_{ii}\phi'(z_i)x_{k'} \le x_j m_{jj}\phi'(z_j)x_{k'}$, and $g_{ik'} \le g_{jk'}$ for each $k' \ne i, j$. Additionally, $x_i m_{ii}\phi'(z_i)x_j \le x_j m_{jj}\phi'(z_j)x_i$, implying $g_{ij} \le g_{ji}$. But then, $x_i = \phi(\sum_{k'\ne i,j} g_{ik'}x_{k'} + g_{ij}x_j) < \phi(\sum_{k'\ne i,j} g_{jk'}x_{k'} + g_{ji}x_i) = x_j$, which is a contradiction. Hence, $x_i \ge x_j$ implies $g_{ik} \ge g_{jk}$ for each $k \ne i, j$.

Therefore, $i \succeq_G j$. Since i and j are taken arbitrarily, G is a GNSG.

Step 3: Let $W(G) \equiv \sum_{i\in N} u_i(\mathbf{x}(G), G)$. Then, $W(G) = \frac{1}{2}\sum_i x_i(G)^2 - \sum_i\sum_j c(g_{ij})$. If G is efficient, then for each $i, j \in N, i \ne j$,

$$\frac{\partial W(G)}{\partial g_{ij}} = \sum_k x_k m_{ki}\phi'(z_i)x_j - c'(g_{ij}) = 0. \tag{13}$$

Since $x_k m_{ki}\phi'(z_i)x_j > 0$ for each $k, i, j \in N$, condition (12) and equation (13) cannot simultaneously hold. Hence, every equilibrium network is inefficient.

Appendix B. Centralities in weighted and directed networks

In the following I contrast link-dominance ordering with all centrality measures considered in König et al. (2014) using examples of GNSGs. The comparison encompasses degree, closeness, eigenvector, Bonacich, and betweenness centralities. Their definitions in the context of weighted and directed networks are adapted from Jackson (2008) and Opsahl et al. (2010). Notably, with directed links, several centrality measures (degree, closeness, and eigenvector) require separate evaluation for each node by distinguishing between an inward measure (e.g., in-degree) and an outward measure (e.g., out-degree). Table 1 presents and compares the various centrality measures for the two networks shown in Fig. 2 and for outward quasi stars in general which are optimal networks under linear link cost. The comparisons show that the inward measure of a centrality can differ significantly from its outward measure and thus at least one of them usually does not align with link-dominance.

- Bonacich centrality: Bonacich centrality is a solution $x = (x_1, \ldots, x_n)$ to the linear system $x = a + bGx$, where $a = (a_1, \ldots, a_n)$ is the base value for each node. When the best-response function is linear such as $x_i = 1 + \lambda\sum_j g_{ij}x_j$, Bonacich centrality exactly equals the equilibrium effort of each agent, with $b = \lambda$ and base value $a_i = 1$ for each i.
- Closeness centrality: In a weighted and directed network, inward closeness centrality for $i \in N$ can be defined as $C_i^{in} = \sum_{j\ne i}\frac{1}{\ell_{ij}}$, where $\ell_{ij} = \min(\frac{1}{g_{ik'}} + \cdots + \frac{1}{g_{kj}})$ is the sum of the inverse of the weights along the path $(j, k, \ldots, k', i)$ and it is minimized over all paths leading from j to i. If there is no path leading from j to i, we set $\ell_{ij} = \infty$ and $\frac{1}{\ell_{ij}} = 0$. The idea is that $\frac{1}{g_{ij}}$ measures the resistance for the directed link from j to i, and $\frac{1}{g_{ik'}} + \cdots + \frac{1}{g_{kj}}$ measures the resistance of the directed path $(j, k, \ldots, k', i)$ from j to i. The minimum $\ell_{ij} = \min(\frac{1}{g_{ik'}} + \cdots + \frac{1}{g_{kj}})$ over all paths from j to i then generalizes the distance measure of the shortest path between two nodes in a unweighted graph. The larger is ℓ_{ij}, the

Table 1
Comparisons of centralities.

	Nodes in Fig. 2a			Nodes in Fig. 2b			Outward quasi-stars		
	1	2	3	1	2	3	1	2	$j \geq 3$
Link-dominance ordering		$\mathbf{1} \succ_G 2 \succ_G 3$			$\mathbf{1} \succ_G 2 \succ_G 3$			$\mathbf{1} \succeq_G 2 \succ_G j$	
Equilibrium effort x_i	1.22	1.26	**1.31**	**1.37**	1.27	1.15		$\mathbf{x_1} \geq x_2 > x_j$	
Aggregate influence α_i	**1.37**	1.27	1.15	1.22	1.26	**1.31**		$\boldsymbol{\alpha_1} \geq \alpha_2 > \alpha_j$	
Bonacich centrality $b = \lambda$, $a = 1$	Same as x_i for linear best-response								
In-degree	0.17	0.21	**0.25**	**0.31**	0.21	0.11	$\mathbf{g_{12}}$	g_{21}	g_{j1}
Out-degree	**0.31**	0.21	0.11	0.17	0.21	**0.25**	$g_{21} + \sum_j g_{j1}{}^*$	$g_{12}{}^*$	0
Inward closeness	0.04	0.04	**0.06**	**0.08**	0.05	0.03	$\mathbf{g_{12}}$	g_{21}	g_{31}
Outward closeness	**0.08**	0.05	0.03	0.04	0.04	**0.06**	$g_{21} + \sum_j g_{j1}{}^*$	$g_{12}{}^*$	0
Betweenness	0	0	0	0	0	0	**1**	0	0
Inward eigenvector centrality	0.50	0.563	**0.66**	**0.73**	0.58	0.36	**0.70**	0.64	0.22
Outward eigenvector centrality	**0.73**	0.58	0.36	0.50	0.56	**0.66**	0.67	**0.74**	0

In computing x_i and α_i for the two graphs in Fig. 2 presented in the main text, I assume $x_i = 1 + \lambda \sum_j g_{ij} x_j$ with $\lambda = 0.1$ and the planner's objective function being $\pi(G) = \sum_i x_i(G)$. The last three columns examine optimal GNSGs under linear link costs $c(g_{ij}) = g_{ij}$, with eigenvector centralities calculated for G_4 in Fig. 5 in the manuscript. Bold text indicates the most central agent based on each centrality measure. The * symbol highlights that the most central agent measured by out-degree centrality and outward closeness can be either agent 1 or agent 2 in an optimal GNSG (last three columns), depending on the planner's objective function.

more distant two nodes are, and the more resistance there is when traveling from one to another. Similarly, outward closeness centrality is defined by $C_i^{out} = 1/\sum_{j\neq} \ell_{ji}$.

- Betweenness centrality: In an unweighted network, betweenness centrality is defined by the fraction of shortest paths that go through the node of focus. In a weighted and directed network, given that the resistance of a path is defined by $\frac{1}{g_{ik'}} + \cdots + \frac{1}{g_{kj}}$ and the "shortest" path from j to i can be defined by the path with the least resistance $\ell_{ij} = \min(\frac{1}{g_{ik'}} + \cdots + \frac{1}{g_{kj}})$, let $P_{ij}(k)$ denote the number of paths from j to i with least resistance (over all paths from j to i), and $P_{ij}(k)$ denote the number of least resistance paths from j to i that go through k. Then, following Opsahl et al. (2010), we can define the betweenness centrality for i by $B_i = \frac{\sum_{k\neq j; i\notin\{k,j\}} P_{kj}(i)}{\sum_{k\neq j; i\notin\{k,j\}} P_{kj}}$; since this definition has considered paths connecting any pair of nodes from both directions, there is no distinction between outward betweenness centrality and inward betweenness centrality.
- Eigenvector centrality: Let $\delta > 0$ be the largest eigenvalue of G. Then inward (right-hand) eigenvector centrality is defined by $e_i^{in} = \sum_{j\neq i} g_{ij} e_j^{in}$, where $e^{in} = (e_1^{in}, \ldots, e_n^{in})$ solves $\delta e^{in} = G e^{in}$. Similarly, onward (left-hand) eigenvector centrality is defined by $e_i^{out} = \sum_{j\neq i} g_{ij} e_j^{out}$, where $e^{out} = (e_1^{out}, \ldots, e_n^{out})$ solves $\delta e^{out} = e^{out} G$.

Examples presented in Table 1 yield two observations. First, degree, closeness, and eigenvector centralities can already yield distinct orderings based on their inward measures and outward measures, even while maintaining the same link-dominance ordering. This suggests that we cannot generally expect inward centrality measures to coincide with their outward counterparts in a generalized nested split graph (GNSG), nor can we expect them to consistently align with link-dominance. Second, according to Theorem 1 in the main text, link-dominance ordering $\succeq_G$ necessarily aligns with the ordering based on equilibrium effort $x_i(G)$ and aggregate influence $\alpha_i(G)$ in a GNSG. Furthermore, the examples show that for optimal GNSGs under linear link cost, centralities of Bonacich, in-degree, inward closeness, and inward eigenvector induce the same ordering as link-dominance. However, out-degree, outward closeness, and outward eigenvector centralities still produce different orderings.

Appendix C. Supplementary material

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.geb.2023.04.010.

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